

Bispectra from two-field inflation using the long-wavelength formalism

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Abstract. We use the long-wavelength formalism to compute the bispectral non-Gaussianity produced in two-field inflation. We find an exact result that is used as the basis of numerical studies, and an explicit analytical slow-roll expression for several classes of potentials that gives insight into the origin and importance of the various contributions to f_{NL} . We also discuss the momentum dependence of f_{NL} . Based on these results we find a simple model that produces a relatively large non-Gaussianity. We show that the long-wavelength formalism is a viable alternative to the standard δN formalism, and can be preferable to it in certain situations.

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1 Introduction

During inflation the energy density of the Universe is assumed to be dominated by the potential energy of one or more scalar fields in order to have a sufficiently rapid expansion to solve the homogeneity problems that plagued pre-inflationary cosmology (horizon, flatness, etc.). Very importantly, inflation also provides the initial adiabatic density perturbation that generated the large-scale structure observed today. Observations of the fluctuations in the cosmic microwave background radiation (CMB), in particular those made by the WMAP satellite, have verified the basic predictions of inflation. The most important observational parameters so far from the point of view of inflation have been the amplitude and the slope (spectral index) of the primordial power spectrum, as well as some limits on the amount of tensor perturbations and the running of the spectral index.

Unfortunately this small number of observational parameters means that a very large number of quite different inflation models are all still consistent with the data. To further narrow down the number of viable inflation models additional observables are required. A very promising candidate is the non-Gaussianity of the primordial power spectrum. In its most simple form this is encoded as a non-zero three-point correlator of the CMB temperature fluctuations, or equivalently a non-zero bispectrum, which is the Fourier (or spherical harmonic on the sphere) transform of the three-point correlator. The quantity defined as the bispectrum divided by the power spectrum squared is called f_{NL} . The current limits on this parameter f_{NL} (assuming momentum dependence of the local type, relevant for this paper) after seven years of WMAP data are $-10 < f_{\text{NL}} < 74$ at 95% confidence level [1]. The newly launched Planck satellite [2] is expected to significantly improve these constraints, down to 1σ error bars of about 3–5 (depending on the use of polarization data) [3, 4]. While standard single-field slow-roll inflation predicts an unobservably small value of f_{NL} [5], many other models predict much larger values that could be detected or ruled out by Planck.

Both supersymmetric particle theory and string theory suggest the existence of multiple scalar fields that can influence the early Universe. If more than a single scalar field plays a role during inflation, isocurvature fluctuations will be produced in addition to the adiabatic one. While these isocurvature fluctuations might have directly observable consequences in the CMB [1], in this paper we are only interested in the effect of the isocurvature fluctuations on the adiabatic one during inflation. This effect can be important even if the isocurvature fluctuations disappear after inflation. The important point here is that while in single-field inflation the adiabatic perturbation is constant on super-horizon scales, this is no longer true in multiple-field inflation. In fact, the isocurvature perturbation acts as a source for the adiabatic perturbation on super-horizon scales and this source is multiplied by the η^\perp parameter [6]. This η^\perp (defined properly in the next section) is proportional to the component of the field acceleration perpendicular to the field velocity. In other words, η^\perp is non-zero if the field trajectory makes a turn in field space. Only during such a turn will the isocurvature mode influence the adiabatic one on super-horizon scales (see also [7]).¹

There are two main ways to produce non-Gaussianity during inflation: during and after horizon crossing of a perturbation mode. Horizon crossing is defined as the moment when the physical wavelength $(k/a)^{-1}$ of the fluctuation becomes equal to the Hubble (or horizon) length H^{-1} , i.e. when $k = aH$ (with k the wave number of the mode, a the scale factor

¹This last statement is strictly only true on a flat field manifold (trivial field metric) with standard kinetic terms. On a curved manifold η^\perp can be non-zero even for a straight field trajectory because of the connection terms in the covariant derivatives, see [8] and [9]. In this paper a trivial field metric will be assumed.

of the Universe and $H = \dot{a}/a$ the Hubble parameter). The first type of non-Gaussianity is produced in all inflation models, but it is unobservably small (i.e. slow-roll suppressed), unless the model contains non-standard kinetic terms (higher derivatives), like DBI inflation [10, 11] for example. In this paper we will not consider those models, but instead focus on the super-horizon type of non-Gaussianity. As is clear from the previous paragraph, super-horizon non-Gaussianity can only be produced in multiple-field inflation models where the field trajectory makes a turn in field space. To compute this type of non-Gaussianity we will make use of the long-wavelength formalism developed by Rigopoulos, Shellard and Van Tent [6, 12, 13], hereafter referred to as RSvT.

The main purpose of this paper is twofold. In the first place we want to further work out, simplify and study the general analytic expression for f_{NL} of RSvT in the case of two fields only. In the second place we want to clarify a few remaining formal issues with the formalism, and compare with an alternative formalism for computing f_{NL} , called the δN formalism [14–18].

As will be shown, the expression for f_{NL} simplifies significantly, although a final integral remains that cannot be done analytically. As a starting point for *numerical* work this expression is very useful though, and it gives a fully exact numerical result since no slow-roll approximation is used for the super-horizon evolution. However, to be able to derive explicit *analytic* results, we have to make use of the slow-roll approximation. Using this approximation we derive explicit analytic results for a number of classes of inflationary potentials. Some of these have been treated before in the literature using the δN formalism, but others are new. In particular we show that models with a potential of the form $W(\phi, \sigma) = \alpha\phi^p + \beta\sigma^q$ with $p = q$ will never give a large f_{NL} that persists until the end of inflation (unless inflation somehow ends right during the turn of the field trajectory, but in that case a very careful treatment of the transition at the end of inflation will be required). We also present a simple model that does produce a “large” f_{NL} of the order of a few. The reason we choose this model is that it can be treated not only numerically, but also analytically.

The formalism of RSvT also allows us to compute the momentum dependence of f_{NL} due to the fact that different modes cross the horizon at different times. This effect has usually been ignored in the literature where it was often assumed that f_{NL} is momentum-independent (see e.g. [19]), although recently people have started looking into this [20, 21]. Here we compute this momentum dependence in an exact way and show that, depending on the model, it can lead to relative effects of order 10% even within the range of momenta that are observable by Planck.

We have extended the formalism of RSvT with an exact treatment of the second-order source term at horizon crossing. While negligibly small in the models we consider, the inclusion of this term allows for an exact analytic comparison with the δN formalism. We find that our analytic slow-roll results agree exactly with those derived using the δN formalism, where available. For models where slow roll breaks down long after horizon crossing and which have to be treated numerically we also find excellent agreement.

Apart from providing an alternative way of computing the bispectrum, which is always useful, the long-wavelength formalism provides a number of advantages compared to the δN formalism. Very importantly, the long-wavelength formalism allows for a simple physical interpretation of the different terms, showing the contributions from adiabatic and isocurvature modes and making clear why some of them can become big and others cannot. While we do not pursue this in the present paper, the formalism also provides the solution for the second-order isocurvature perturbation and hence the isocurvature bispectrum could be

computed as easily as the adiabatic one.

Many people have worked on non-Gaussianity, both predictions from inflation and estimators for CMB observations. The reduced bispectrum has been given in [22] for the equilateral type, in [3, 4] for the local type and in [23] for the orthogonal type of the primordial bispectrum. The different shapes were studied in detail in [24, 25]. Bispectrum estimators were developed in [22, 26–29]. All kinds of inflationary models have been studied as well. For instance, we have learned that single-field inflation models cannot produce large non-Gaussianity [5], unless some non-trivial potential is used [30] or higher derivative contributions are introduced as for the Dirac-Born-Infeld action [10, 31–33] or K-inflation [34, 35]. The study of the effective theory of inflation has also turned out to be very fruitful [36]. These models were extended to incorporate multiple fields in [11, 32, 37–39]. Large non-Gaussianity can also be produced at the end of inflation [40–45] or after inflation, in models with varying inflaton decay rate [46] and in curvaton models [47–52].

Large scale evolution of perturbations during inflation up to second order became possible through their consistent gauge invariant definition [53–55]. Within the δN formalism several authors have investigated the bispectra of specific multiple field inflation models [56–61]. Two-field models, being easier to deal with, have gained popularity though. Vernizzi and Wands studied the double field sum potential [19], while the double product potential was studied in [62]. Conditions for large non-Gaussianity were found in [63].

The paper is organized as follows. In section 2 we summarize the long-wavelength formalism of RSvT and define the various quantities used in the paper. Here we also describe the second-order source term, which is a new extension of the formalism. At the end of the section we give a brief overview of the δN formalism for comparison purposes. In section 3 we work out the general expression for f_{NL} in the case of two fields. We also compute the momentum dependence that arises in the case that not all scales cross the horizon at the same time. These expressions, derived without using any super-horizon slow-roll approximation, are one of the main results of the paper and the starting point for our numerical analyses. In order to find completely explicit analytic expressions for f_{NL} , however, we do need to assume slow roll, as well as some conditions on the potential. This is treated in section 4, where we also compare our analytic results with those obtained using the δN formalism, for as far as the latter exist. In section 5 we use the two-field quadratic potential in order to compare our exact numerical results with those of the δN -formalism. We also present a simple potential that can produce an f_{NL} of the order of a few, which falls into the category of potentials that can be treated analytically, thus allowing us to test our results. We conclude in section 6. Finally, in the appendices we give supplementary information on the basis in field space that we use, compute the second-order source term, comment on some gauge and formal issues, and provide several intermediate steps of our calculations. Note that A introduces in particular a small improvement of the basis defined in [8] that makes it more convenient for numerical calculations during periods when the fields oscillate.

2 Basic equations and definitions

This section sets up the starting point for the work in the following sections. It is mostly a summary of the long-wavelength formalism and its results as presented in [6], although the part about the second-order source term in section 2.1 and some results on the Green’s functions in section 2.2 are new. Section 2.1 describes the non-linear equations for the perturbations, section 2.2 shows how to solve them using Green’s functions, and section 2.3 gives

the formal expressions for the two and three point correlation functions of the perturbations, or rather their Fourier transforms, the power spectrum and the bispectrum. It is the latter that will be worked out in great detail in the rest of the paper. Finally in section 2.4 we give a brief overview of the δN formalism for comparison purposes later in the paper.

2.1 Non-linear equations

In the long-wavelength formalism space-time is described by the long-wavelength metric [64]

$$ds^2 = -N^2(t, \mathbf{x})dt^2 + a^2(t, \mathbf{x})d\mathbf{x}^2, \quad (2.1)$$

where the lapse function $N(t, \mathbf{x})$ defines the time slicing and $a(t, \mathbf{x})$ is the space dependent scale factor. The Hubble parameter is defined as $H \equiv \partial_t \ln a / N$. In order to simplify superhorizon calculations we choose to work in a flat gauge $NH = 1$, that is we choose time slices in which the expansion of the universe is homogeneous and the time variable coincides with the number of e-folds $t = \ln a$.

On the matter side we assume in this paper two scalar fields with a trivial field metric, although the formalism can in principle deal with an arbitrary number of scalar fields living on an arbitrary field manifold. The energy-momentum tensor for the two fields ϕ^A ($A, B = 1, 2$) is

$$T_{\mu\nu} = \delta_{AB} \partial_\mu \phi^A \partial_\nu \phi^B - g_{\mu\nu} \left(\frac{1}{2} \delta_{AB} \partial^\lambda \phi^A \partial_\lambda \phi^B + W \right), \quad (2.2)$$

where W is the potential. The Einstein summation convention is assumed throughout this paper. We also define the derivative of the fields with respect to proper time as $\Pi^A \equiv \partial_t \phi^A / N$, with length Π .

The long-wavelength formalism corresponds to the leading-order approximation of the spatial gradient expansion (see [65, 66] and references therein). In this expansion all quantities are expanded in terms of a small parameter $1/(HL)$, where L is the characteristic physical length scale of the perturbations (i.e. proportional to a). The leading-order approximation of the spatial gradient expansion is equivalent to neglecting the k^2 term (which comes from the second-order spatial gradient and is of order $\mathcal{O}(1/(HL)^2)$) with respect to the $\mathcal{O}(1)$ terms in the equation for the perturbation modes. Because of the very rapid growth of a during inflation, this is in principle a well-justified approximation from just a few e-folds after horizon crossing of the perturbation mode under consideration, when the decaying mode will have disappeared. However, as pointed out in [67, 68], if slow roll is broken at horizon crossing and for some e-folds afterwards, a cancellation of the $\mathcal{O}(1)$ terms can cause the decaying mode to remain important during this period. In those papers it was shown that for single-field inflation there may be an enhancement of the curvature perturbation both at first and second order due to the effect of the k^2 term even on super-horizon scales, if the decaying mode has not yet vanished. In this paper we will assume slow roll to hold around horizon crossing, so that the decaying mode will quickly disappear, and the long-wavelength approximation is valid on super-horizon scales.

The Einstein and field equations are given by [12]

$$\begin{aligned} H^2 &= \frac{\kappa^2}{3} \left(\frac{\Pi^2}{2} + W \right), & \dot{H} &= -\frac{\kappa^2 \Pi^2}{2H}, \\ \dot{\Pi}^A &= -3\Pi^A - \frac{W_{,A}}{H}, & \partial_i H &= -\frac{\kappa^2}{2} \Pi_A \partial_i \phi^A, \end{aligned} \quad (2.3)$$

with $\kappa^2 \equiv 8\pi G = 8\pi/m_{pl}^2$ and $W_{,A} \equiv \partial W/\partial\phi^A$. While the first three equations look like background equations, they are actually fully non-linear since $H(t, \mathbf{x})$, $\phi^A(t, \mathbf{x})$, and $\Pi^A(t, \mathbf{x})$ defined above are functions of both time and space. We define an orthonormal basis e_m^A in field space (see A for details) through the field velocity, $e_1^A \equiv \Pi^A/\Pi$, and successively higher-order time derivatives of the fields [8]. This basis allows us to easily distinguish effectively single-field effects with $m = 1$ from truly multiple-field effects with $m \geq 2$. The local slow-roll parameters then take the form

$$\begin{aligned} \epsilon(t, \mathbf{x}) &\equiv -\frac{\dot{H}}{H}, & \eta^\parallel(t, \mathbf{x}) &\equiv \frac{e_{1A}\dot{\Pi}^A}{\Pi}, & \eta^\perp(t, \mathbf{x}) &\equiv \frac{e_{2A}\dot{\Pi}^A}{\Pi}, \\ \chi(t, \mathbf{x}) &\equiv \tilde{W}_{22} + \epsilon + \eta^\parallel, & \xi^\parallel(t, \mathbf{x}) &\equiv \frac{e_{1A}\ddot{\Pi}^A}{\Pi} - \epsilon\eta^\parallel, & \xi^\perp(t, \mathbf{x}) &\equiv \frac{e_{2A}\ddot{\Pi}^A}{\Pi} - \epsilon\eta^\perp, \end{aligned} \quad (2.4)$$

where $\tilde{W}_{mn} \equiv W_{mn}/(3H^2)$ and $W_{mn} \equiv e_m^A e_n^B W_{,AB}$. Throughout this paper the indices l, m, n will indicate components in the basis defined above, taking the values 1 and 2. The correction term in the expressions for ξ^\parallel and ξ^\perp comes about because the proper definition is $\xi^A \equiv ([(1/N)\partial_t]^2 \Pi^A)/(H^2 \Pi)$ and $NH = 1$. We note here that we have not made any slow-roll approximations so far; the above quantities should be viewed as short-hand notation and can be large. We also give the time derivatives of the slow-roll parameters,

$$\begin{aligned} \dot{\epsilon} &= 2\epsilon(\epsilon + \eta^\parallel), & \dot{\eta}^\parallel &= \xi^\parallel + (\eta^\perp)^2 + (\epsilon - \eta^\parallel)\eta^\parallel, & \dot{\eta}^\perp &= \xi^\perp + (\epsilon - 2\eta^\parallel)\eta^\perp, \\ \dot{\chi} &= \epsilon\eta^\parallel + 2\epsilon\chi - (\eta^\parallel)^2 + 3(\eta^\perp)^2 + \xi^\parallel + \frac{2}{3}\eta^\perp\xi^\perp + \tilde{W}_{221}, \end{aligned} \quad (2.5)$$

and of the unit vectors,

$$\dot{e}_1^A = \eta^\perp e_2^A, \quad \dot{e}_2^A = -\eta^\perp e_1^A. \quad (2.6)$$

Here we have defined $\tilde{W}_{lmn} \equiv (\sqrt{2\epsilon}/\kappa)W_{lmn}/(3H^2)$. Hence the tilde has a different meaning in the case of two and of three derivatives, but it is these specific combinations that always appear in the equations.

Our main variable describing the perturbations is [12]

$$\zeta_i^m \equiv \delta_{m1}\partial_i \ln a - \frac{\kappa}{\sqrt{2\epsilon}} (e_{mA}\partial_i \phi^A), \quad (2.7)$$

with $m = 1$ the adiabatic component and $m = 2$ the isocurvature one. The non-linear quantity ζ_i^m has been constructed to transform as a scalar under changes of time slicing on long wavelengths. In the single-field case and when linearized it is just the spatial gradient of the well-known curvature perturbation ζ . Of course in the gauge we have chosen $\partial_i \ln a = 0$ and the first term in the expression for ζ_i^m disappears. The exact evolution equations for ζ_i^m and its time derivative $\theta_i^m \equiv \partial_t (\zeta_i^m)$ are [6]

$$\dot{v}_{ia}(t, \mathbf{x}) + A_{ab}(t, \mathbf{x})v_{ib}(t, \mathbf{x}) = 0, \quad \text{where } v_{ia} \equiv (\zeta_i^1, \zeta_i^2, \theta_i^2), \quad (2.8)$$

so $a, b = 1, 2, 3$. We have simplified the system by omitting the time derivative of the adiabatic component, θ_i^1 , since it is given by $\theta_i^1 = 2\eta^\perp \zeta_i^2$, valid fully non-linearly [6]. The matrix A_{ab} in the two-field case is

$$\mathbf{A} = \begin{pmatrix} 0 & -2\eta^\perp & 0 \\ 0 & 0 & -1 \\ 0 & 3\chi + 2\epsilon^2 + 4\epsilon\eta^\parallel + 4(\eta^\perp)^2 + \xi^\parallel & 3 + \epsilon + 2\eta^\parallel \end{pmatrix}. \quad (2.9)$$

To solve this equation, we expand the system as an infinite hierarchy of linear inhomogeneous perturbation equations. To first and second order we obtain

$$\dot{v}_{ia}^{(1)} + A_{ab}^{(0)}(t)v_{ib}^{(1)} = b_{ia}^{(1)}(t, \mathbf{x}), \quad (2.10)$$

$$\dot{v}_{ia}^{(2)} + A_{ab}^{(0)}(t)v_{ib}^{(2)} = -A_{ab}^{(1)}(t, \mathbf{x})v_{ib}^{(1)} + b_{ia}^{(2)}(t, \mathbf{x}). \quad (2.11)$$

Here the source terms $b^{(1)}$ and $b^{(2)}$ have been added to describe the influence of the short-wavelength modes on the long-wavelength system given in (2.8), providing the necessary initial conditions. $A_{ab}^{(1)}$ is found by perturbing the exact \mathbf{A} matrix, giving $A_{ab}^{(1)}(t, \mathbf{x}) = \bar{A}_{abc}^{(0)}(t)v_c^{(1)}(t, \mathbf{x})$. The explicit form of $\bar{\mathbf{A}}$ is given in (D.6), where we have dropped the superscript (0) for notational convenience. We have also defined $v_c^{(1)} \equiv \partial^{-2}\partial^i v_{ic}^{(1)}$.

The source term $b_{ia}^{(1)}$ can be expressed in terms of the linear mode function solutions $X_{am}^{(1)}$, using a window function $\mathcal{W}(k)$ which guarantees that short wavelengths are cut out to get only the contribution to the long-wavelength system,

$$b_{ia}^{(1)} = \int \frac{d^3\mathbf{k}}{(2\pi)^{3/2}} \dot{\mathcal{W}}(k) X_{am}^{(1)}(k) \hat{a}_m^\dagger(\mathbf{k}) i k_i e^{i\mathbf{k}\cdot\mathbf{x}} + \text{c.c.} \quad (2.12)$$

The quantum creation (\hat{a}_m^\dagger) and conjugate annihilation (\hat{a}_m) operators satisfy the usual commutation relations. The linear mode solutions can be determined exactly numerically, or analytically within the slow-roll approximation (which, as observations indicate, seems to be a very good approximation at horizon crossing). See section 2.2 for explicit expressions for these $X_{am}^{(1)}$ as well as for the window function \mathcal{W} .

The source term $b_{ia}^{(2)}$ was either neglected by RSvT [6] because it is small, or in earlier papers (e.g. [13]) approximated by perturbing $X_{am}^{(1)}$, which turned out not to be a good approximation. While this contribution to f_{NL} is indeed small, here we compute it explicitly in order to allow for an exact comparison with known results in the literature. We find that it can be expressed by means of the window function as

$$\begin{aligned} b_{ia}^{(2)} = & \int \frac{d^3\mathbf{k}}{(2\pi)^{3/2}} \int \frac{d^3\mathbf{k}'}{(2\pi)^{3/2}} \dot{\mathcal{W}}(\max(k', k)) \\ & \times \left\{ L_{abc}(t) X_{bm}^{(1)}(k', t) X_{cn}^{(1)}(k, t) \hat{a}_m^\dagger(\mathbf{k}') \hat{a}_n^\dagger(\mathbf{k}) i(k'_i + k_i) e^{i(\mathbf{k}' + \mathbf{k})\cdot\mathbf{x}} \right. \\ & \left. + N_{abc}(t) X_{bm}^{(1)}(k', t) X_{cn}^{(1)}(k, t) \hat{a}_m^\dagger(\mathbf{k}') \hat{a}_n^\dagger(\mathbf{k}) i k_i e^{i(\mathbf{k}' + \mathbf{k})\cdot\mathbf{x}} + \text{c.c.} \right\}, \quad (2.13) \end{aligned}$$

where the derivative of the window function peaks at the scale that exits the horizon last. We have split $b_{ia}^{(2)}$ into a local part proportional to L_{abc} and a non-local part proportional to N_{abc} . In order to find these factors, we generalize the results of Maldacena [5] to multiple fields. This is similar to the work done in [56], but due to the different definitions and gauge choices used in that paper, we found it easier to rederive the results from scratch. Maldacena computed the third-order action for ζ in the uniform energy density gauge. In order to calculate the three-point correlation function he performed a redefinition of ζ to remove terms in the action which are proportional to the equations of motion. Our generalization of this calculation can be found in B, while the explicit expressions for the components of L_{abc} and N_{abc} are given in section 2.2.

2.2 Green's functions

Equations (2.10) and (2.11), together with the initial condition $v_{ia}(t \rightarrow -\infty) = 0$, can be solved using a simple Green's function $G_{ab}(t, t')$. In matrix notation it satisfies [6, 13]

$$\frac{d}{dt}\mathbf{G}(t, t') + \mathbf{A}(t)\mathbf{G}(t, t') = \mathbf{0}, \quad \mathbf{G}(t, t) = \mathbf{1}. \quad (2.14)$$

Starting from this equation, to lighten the notation, when we write \mathbf{A} we actually mean $\mathbf{A}^{(0)}$, i.e. the matrix in (2.9) with all local slow-roll parameters replaced by their background version that depends on time only. Looking at this equation of motion and its initial condition, we see that the solution can be written as

$$\mathbf{G}(t, t') = \mathbf{F}(t)\mathbf{F}^{-1}(t'), \quad (2.15)$$

where $\mathbf{F}(t)$ satisfies the same equation of motion (2.14) as $\mathbf{G}(t, t')$ with an arbitrary initial condition. From this we immediately derive that

$$\frac{d}{dt'}\mathbf{G}(t, t') - \mathbf{G}(t, t')\mathbf{A}(t') = \mathbf{0}. \quad (2.16)$$

The solution of (2.10) and (2.11) can now be written as the time integral of G_{ab} contracted with the terms on the right-hand side of these equations:

$$v_{ia}^{(1)}(t, \mathbf{x}) = \int \frac{d^3\mathbf{k}}{(2\pi)^{3/2}} v_{am}(k, t) \hat{a}_m^\dagger(\mathbf{k}) i k_i e^{i\mathbf{k}\cdot\mathbf{x}} + \text{c.c.}, \quad (2.17)$$

with

$$v_{am}(k, t) = \int_{-\infty}^t dt' G_{ab}(t, t') \dot{W}(k, t') X_{bm}^{(1)}(k, t') \quad (2.18)$$

and

$$v_{ia}^{(2)}(t, \mathbf{x}) = - \int_{-\infty}^t dt' G_{ab}(t, t') \bar{A}_{bcd}(t') v_{ic}^{(1)}(t', \mathbf{x}) v_d^{(1)}(t', \mathbf{x}) + \int_{-\infty}^t dt' G_{ab}(t, t') b_{ib}^{(2)}(t', \mathbf{x}). \quad (2.19)$$

As before, $v_d^{(1)} \equiv \partial^{-2} \partial^i v_{id}^{(1)}$.

Written in components (2.14) gives the following equations for the Green's functions:

$$\begin{aligned} \frac{d}{dt} G_{1x}(t, t') &= 2\eta^\perp(t) G_{2x}(t, t'), \\ \frac{d}{dt} G_{2x}(t, t') &= G_{3x}(t, t'), \\ \frac{d}{dt} G_{3x}(t, t') &= -A_{32}(t) G_{2x}(t, t') - A_{33}(t) G_{3x}(t, t'), \\ G_{ab}(t, t) &= \delta_{ab}. \end{aligned} \quad (2.20)$$

We can also rewrite this as a second-order differential equation for G_{2x} :

$$\frac{d^2}{dt^2} G_{2x}(t, t') + A_{33}(t) \frac{d}{dt} G_{2x}(t, t') + A_{32}(t) G_{2x}(t, t') = 0. \quad (2.21)$$

For the derivatives with respect to t' we find:

$$\begin{aligned}\frac{d}{dt'} G_{x2}(t, t') &= -2\eta^\perp(t')\delta_{x1} + A_{32}(t')G_{x3}(t, t'), \\ \frac{d}{dt'} G_{x3}(t, t') &= -G_{x2}(t, t') + A_{33}(t')G_{x3}(t, t').\end{aligned}\quad (2.22)$$

The solutions for the $x = 1$ components of (2.20) are simple: $G_{11} = 1$, $G_{21} = G_{31} = 0$. To find the solutions for the $x = 2, 3$ components we assume that we have found a solution $g(t)$ that satisfies (2.21). Then a second, independent, solution is given by

$$f(t) = g(t) \int^t d\bar{t} Y(\bar{t}), \quad Y(t) \equiv \frac{1}{g^2(t)} e^{-\int^t d\bar{t} A_{33}(\bar{t})} = \frac{1}{g^2(t)} \frac{e^{-3t}}{H(t)\epsilon(t)}. \quad (2.23)$$

Hence

$$G_{23}(t, t') = \frac{1}{g(t')Y(t')} f(t) - \frac{f(t')}{g^2(t')Y(t')} g(t), \quad (2.24)$$

$$G_{22}(t, t') = \left(\frac{\dot{g}(t')f(t')}{g^3(t')Y(t')} + \frac{1}{g(t')} \right) g(t) - \frac{\dot{g}(t')}{g^2(t')Y(t')} f(t) = \frac{g(t)}{g(t')} - \frac{\dot{g}(t')}{g(t')} G_{23}(t, t'), \quad (2.25)$$

and

$$G_{33}(t, t') = \frac{\dot{g}(t)}{g(t)} G_{23}(t, t') + \frac{g(t)Y(t)}{g(t')Y(t')}, \quad G_{32}(t, t') = \frac{\dot{g}(t)}{g(t)} G_{22}(t, t') - \frac{\dot{g}(t')}{g(t')} \frac{g(t)Y(t)}{g(t')Y(t')}. \quad (2.26)$$

Of course $G_{13}(t, t') = 2 \int_{t'}^t d\bar{t} \eta^\perp(\bar{t}) G_{23}(\bar{t}, t')$ and $G_{12}(t, t') = 2 \int_{t'}^t d\bar{t} \eta^\perp(\bar{t}) G_{22}(\bar{t}, t')$. For exact calculations the Green's functions will be determined numerically, but in an approximate slow-roll treatment we can sometimes find analytic solutions, see section 4.

For the linear mode solutions at horizon crossing, $X_{am}^{(1)}$, we will assume in this paper the analytic slow-roll solutions determined in [8]. Observations of the spectral index indicate that slow roll is a good approximation *at* horizon crossing. Note however that, with the exception of section 4, we do not assume slow roll to hold *after* horizon crossing. Moreover, the assumption of slow roll at horizon crossing is not a requirement to compute these linear solutions, we could just as well numerically compute the linear mode solutions exactly. For the window function used in the calculation of the linear solution we take a step function, see [6, 12], so that its time derivative is a delta function: $\dot{\mathcal{W}} = \delta(kc/(aH\sqrt{2}) - 1)$, where c is a constant of the order of a few, e.g. $c = 3$. Then

$$v_{am}(t) = G_{ab}(t, t_*) X_{bm}^{(1)}(t_*) \Theta(t - t_*), \quad (2.27)$$

where the step function $\Theta(x)$ equals 1 for $x \geq 0$ and 0 for $x < 0$. The time t_* is defined by $aH = kc/\sqrt{2}$, i.e. a time slightly after horizon crossing when we have entered the long-wavelength regime. While results right at t_* of course depend on the details of the window function, a few e-folds later any dependence on \mathcal{W} has disappeared. Moreover, under the assumption of slow roll at horizon crossing, all quantities change very little between horizon crossing and t_* , so that final results do not depend on the choice of c and t_* can be taken equal to the horizon-crossing time determined from $k = aH$ in the final expressions. Defining γ_* as $\gamma_* \equiv -\kappa H_*/(2k^{3/2}\sqrt{\epsilon_*})$, where the subscript $*$ means evaluation at $t = t_*$, the matrix

$\mathbf{X}^{(1)}(t_*)$ is given by $X_{11}^{(1)}(t_*) = X_{22}^{(1)}(t_*) = \gamma_*$, $X_{32}^{(1)}(t_*) = -\chi_*\gamma_*$, the other components being zero [6]. Hence we have

$$\begin{aligned} v_{11} &= \gamma_* \Theta(t - t_*), \quad v_{12}(t) = \gamma_* (G_{12}(t, t_*) - \chi_* G_{13}(t, t_*)) \Theta(t - t_*), \\ v_{21} &= 0, \quad v_{22}(t) = \gamma_* (G_{22}(t, t_*) - \chi_* G_{23}(t, t_*)) \Theta(t - t_*), \\ v_{31} &= 0, \quad v_{32}(t) = \gamma_* (G_{32}(t, t_*) - \chi_* G_{33}(t, t_*)) \Theta(t - t_*). \end{aligned} \quad (2.28)$$

We also define the short-hand notation \bar{v}_{am} by $v_{am}(t) = \gamma_* \Theta(t - t_*) \bar{v}_{am}(t)$.

For the second-order horizon-crossing solutions we find from (B.10) and (B.11) (see B) that the slow-roll matrices L_{abc} and N_{abc} have elements satisfying

$$\begin{aligned} L_{111*} &= \epsilon_* + \eta_*^\parallel, & L_{122*} &= -(\epsilon_* + \eta_*^\parallel - \chi_*), \\ L_{211*} &= \eta_*^\perp, & L_{222*} &= \eta_*^\perp, \\ L_{112*} + L_{121} &= 2\eta_*^\perp, & N_{112*} + N_{121*} &= -2\eta_*^\perp, \\ L_{212*} + L_{221*} &= 2(\epsilon_* + \eta_*^\parallel - \chi_*), & N_{212*} + N_{221*} &= \chi_*, \end{aligned} \quad (2.29)$$

with the other elements of N_{abc} being zero. As explained in the appendix, a slow-roll approximation which expresses θ^2 in terms of ζ^2 has been used. This means in particular that the subscripts a, b, c only take the values 1 and 2, but not 3. However, for consistency in the notation, we will define here all entries of L_{abc} and N_{abc} to be equal to zero if one or more of the indices are equal to 3.

2.3 Two and three point statistics

So far we have used time slices on which the expansion of the universe is homogeneous ($\partial_i \ln a = 0$), since it simplifies super-horizon calculations. However, to make contact with the proper gauge-invariant expression for ζ it turns out to be necessary to change to uniform energy density time slices ($\partial_i \rho = 0$). On such slices the adiabatic perturbation variable has the simple form

$$\tilde{\zeta}_i^1 = \partial_i \ln a. \quad (2.30)$$

At first and second order the relation between the adiabatic component of ζ_i in the two gauges is (see [6] and the discussion in C)

$$\tilde{\zeta}_i^{(1)1}(t) = \zeta_i^{(1)1}(t), \quad (2.31)$$

$$\tilde{\zeta}_i^{(2)1}(t) = \zeta_i^{(2)1}(t) + 2\eta^\perp \zeta^{(1)1} \zeta_i^{(1)2}, \quad (2.32)$$

where again $\zeta^{(1)1} \equiv \partial^{-2} \partial^i \zeta_i^{(1)1}$. Indeed one can show that not only do we end up with a total gradient through this gauge transformation, we also obtain the gauge-invariant quantity corresponding to the curvature perturbation ζ : in the flat gauge and for superhorizon scales one can show that [69] (see also [55] for the energy density definition of ζ_i)

$$\zeta_i^{(2)1} = \partial_i \zeta^{(2)1} - \zeta^{(1)1} \dot{\zeta}_i^{(1)1}. \quad (2.33)$$

The second term on the right-hand side cancels exactly the gauge transformation term and we are left with the space gradient of the gauge-invariant quantity.

From (2.17) we see that the Fourier coefficients of $\zeta^{(1)1}(\mathbf{x}, t)$ are given by $\zeta_{\mathbf{k}}^{(1)1}(t) = v_{1m}(k, t)(\hat{a}_m^\dagger(\mathbf{k}) + \hat{a}_m(-\mathbf{k}))$. Hence the power spectrum, that is the two-point correlator of the Fourier coefficients, is

$$\langle \tilde{\zeta}_{\mathbf{k}_1}^{(1)1} \tilde{\zeta}_{\mathbf{k}_2}^{(1)1} \rangle = \delta^3(\mathbf{k}_1 + \mathbf{k}_2) v_{1m}(k_1, t) v_{1m}(k_2, t). \quad (2.34)$$

Conventionally (see e.g. [70, 71]) a quantity \mathcal{P}_ζ , also called the power spectrum of ζ , is defined to remove the overall delta function and the factor $1/k^3$ coming from the v_{1m} (see (2.28)), as follows:

$$\mathcal{P}_\zeta(k, t) \equiv \frac{k^3}{2\pi^2} v_{1m}(k, t) v_{1m}(k, t). \quad (2.35)$$

The scalar spectral index is then defined as

$$n_\zeta - 1 \equiv \frac{d \ln \mathcal{P}_\zeta}{d \ln k} = \frac{d \ln \mathcal{P}_\zeta}{dt_*} \frac{dt_*}{d \ln k} = \frac{d \ln \mathcal{P}_\zeta}{dt_*} \frac{1}{1 - \epsilon_*}, \quad (2.36)$$

where we used that $k = aH\sqrt{2}/c$ and $\dot{H} = -\epsilon H$ for our gauge where $t = \ln a$.

By combining the different permutations of $\langle \tilde{\zeta}_{\mathbf{k}_1}^{(2)1} \tilde{\zeta}_{\mathbf{k}_2}^{(1)1} \tilde{\zeta}_{\mathbf{k}_3}^{(1)1} \rangle$ of the Fourier components of the linear and second-order adiabatic solutions (first subtracting the average of $\tilde{\zeta}^{(2)1}(\mathbf{x}, t)$ to get rid of the divergent part), we find the bispectrum² [6]

$$\begin{aligned} \langle \tilde{\zeta}_{\mathbf{k}_1}^1 \tilde{\zeta}_{\mathbf{k}_2}^1 \tilde{\zeta}_{\mathbf{k}_3}^1 \rangle^{(2)} &= (2\pi)^{-3/2} \delta^3 \left(\sum_s \mathbf{k}_s \right) [f(k_1, k_2) + f(k_1, k_3) + f(k_2, k_3)] \\ &\equiv (2\pi)^{-3/2} \delta^3 \left(\sum_s \mathbf{k}_s \right) B_\zeta(k_1, k_2, k_3), \end{aligned} \quad (2.37)$$

where

$$\begin{aligned} f(k, k') &\equiv v_{1m}(k) v_{1n}(k') \left(\eta^\perp v_{2m}(k) v_{1n}(k') + \frac{1}{2} G_{1a}(t, t_{k'}) M_{abc}(t_{k'}) X_{bm}(k, t_{k'}) X_{cn}(k', t_{k'}) \right. \\ &\quad \left. - \frac{1}{2} \int_{-\infty}^t dt' G_{1a}(t, t') \bar{A}_{abc} v_{bm}(k) v_{cn}(k') \right) + k \leftrightarrow k', \end{aligned} \quad (2.38)$$

where k' refers to the scale that exits the horizon last and $M_{abc} \equiv L_{abc} + N_{abc}$. Finally we introduce the parameter f_{NL} , basically defined as the bispectrum divided by the power spectrum squared, which gives a relative measure of the importance of non-Gaussianities of the bispectral type (see e.g. [5, 19]):

$$\begin{aligned} -\frac{6}{5} f_{\text{NL}} &\equiv \frac{B_\zeta(k_1, k_2, k_3)}{\frac{2\pi^2}{k_1^3} \mathcal{P}_\zeta(k_1) \frac{2\pi^2}{k_2^3} \mathcal{P}_\zeta(k_2) + (k_2 \leftrightarrow k_3) + (k_1 \leftrightarrow k_3)} \\ &= \frac{f(k_1, k_2) + f(k_1, k_3) + f(k_2, k_3)}{v_{1m}(k_1) v_{1m}(k_1) v_{1n}(k_2) v_{1n}(k_2) + 2 \text{ perms.}}. \end{aligned} \quad (2.39)$$

The quotient is called $-\frac{6}{5} f_{\text{NL}}$ and not simply f_{NL} because it was originally defined in terms of the gravitational potential Φ and not ζ as $\Phi = \Phi_L + f_{\text{NL}} (\Phi_L^2 - \langle \Phi_L^2 \rangle)$ [3]. During recombination (matter domination) the two are related by $\zeta = -\frac{5}{3} \Phi$. Moreover, when computing

²In the literature (e.g. [5]) one often sees a factor $(2\pi)^3$ in front of the bispectrum (as well as in front of the power spectrum). This is due to a different definition of the Fourier transform. We use the convention where both the Fourier transform and its inverse have a factor $(2\pi)^{-3/2}$.

the bispectrum divided by the three permutations of the power spectrum squared using this expression of Φ one obtains $2f_{\text{NL}}$ due to the two ways the two Φ_L inside the second-order solution can be combined with the two linear solutions to create the power spectrum. Together these two effects explain the factor $-6/5$.³

2.4 δN -formalism

An alternative formalism to compute f_{NL} is the so-called δN -formalism [14–18]. In order to compare our results of the next sections to those obtained using the δN formalism, for those cases where the latter are available, we give here a brief overview.

The δN formalism uses the fact that the adiabatic perturbation ζ^1 on large scales is equal to the perturbation of the number of e-folds $\delta N(t, t_*)$ between an initial flat hypersurface at $t = t_*$, which is usually taken to be the horizon crossing time, and a final uniform density hypersurface at t . One can then expand the number of e-folds in terms of the perturbations of the fields and their momenta on the initial flat hypersurface

$$\delta N(t, t_*) = \frac{\partial N}{\partial \phi_*^A} \delta \phi_*^A + \frac{\partial N}{\partial \Pi_*^A} \delta \Pi_*^A + \frac{1}{2} \frac{\partial^2 N}{\partial \phi_*^A \partial \phi_*^B} \delta \phi_*^A \delta \phi_*^B + \dots \quad (2.40)$$

So instead of integrating the evolution of ζ^1 through equations (2.10) and (2.11) one can evaluate the derivatives of the number of e-folds at horizon crossing and thus calculate ζ^1 .

Because of the computational difficulty associated with the derivatives with respect to Π^A , slow roll is assumed at horizon exit so that the terms involving the momentum of the fields can be ignored. This is a crucial assumption for the δN formalism. The final formula then reads

$$\delta N(t, t_*) = \frac{\partial N}{\partial \phi_*^A} \delta \phi_*^A + \frac{1}{2} \frac{\partial^2 N}{\partial \phi_*^A \partial \phi_*^B} \delta \phi_*^A \delta \phi_*^B, \quad (2.41)$$

up to second order. From it one finds the following expression for the bispectrum:

$$\langle \tilde{\zeta}_{\mathbf{k}_1}^1 \tilde{\zeta}_{\mathbf{k}_2}^1 \tilde{\zeta}_{\mathbf{k}_3}^1 \rangle^{(2)} = \frac{1}{2} N_{,A} N_{,B} N_{,CD} \langle \delta \phi_{\mathbf{k}_1}^A \delta \phi_{\mathbf{k}_2}^B (\delta \phi^C \star \delta \phi^D)_{\mathbf{k}_3} \rangle + \text{perms.}, \quad (2.42)$$

where \star denotes a convolution and the average of $(\delta \phi \star \delta \phi)$ has been subtracted to avoid divergences. $N_{,A}$ denotes the derivative of N with respect to the field ϕ_*^A . Using Wick's theorem this can be rewritten as products of two-point correlation functions to yield finally

$$-\frac{6}{5} f_{\text{NL}, \delta N} = \frac{N_{,A} N_{,B} N_{,AB}}{(N_{,C} N_{,C})^2}. \quad (2.43)$$

Notice that this result is momentum independent and local in real space, although attempts to generalize to a scale-dependent situation have recently been made in [20, 21]. This formula can be used numerically or analytically to calculate f_{NL} . However, for any analytical results and insight one must assume the slow-roll approximation to hold at all times after horizon exit (see for example [19, 59, 62]), except for the special case of a separable Hubble parameter [72, 73].

³In [6] and earlier papers we used a slightly different definition of f_{NL} which was larger by a factor of $-18/5$. Here we conform to the definition that is now generally accepted in the literature.

3 General analytic expression for f_{NL} for two fields

In this section we will further work out the exact long-wavelength expression for f_{NL} for the case of two fields, given in (2.39). No slow-roll approximation is used on super-horizon scales in this section. In particular this means the formalism can deal with sharp turns in the field trajectory after horizon crossing during which slow roll temporarily breaks down. In the first subsection we restrict ourselves to the case where $k_1 = k_2 = k_3$ to lighten the notation. In the second subsection we show how the result for f_{NL} changes in the case of arbitrary momenta.

3.1 Equal momenta

In the case of equal momenta, equation (2.39) reduces to

$$-\frac{6}{5}f_{\text{NL}} = \frac{-v_{1m}(t)v_{1n}(t)}{(v_{1m}(t)v_{1m}(t))^2} \left\{ \int_{-\infty}^t dt' G_{1a}(t, t') \bar{A}_{abc}(t') v_{bm}(t') v_{cn}(t') - 2\eta^\perp(t) v_{2m}(t) v_{1n}(t) \right. \\ \left. - G_{1a}(t, t_*) M_{abc*} v_{bm}(t_*) v_{cn}(t_*) \right\}. \quad (3.1)$$

We remind the reader that indices l, m, n take the values 1 and 2 (components in the two-field basis), while indices a, b, c, \dots take the values 1, 2, and 3 (labeling the ζ^1 , ζ^2 , and θ^2 components). To make the expressions a bit shorter, we will drop the time arguments inside the integrals, but remember that for the Green's functions the integration variable is the second argument. Using the result (D.1) proved in D.1 we can write \bar{A}_{ab1} as a time derivative and do an integration by parts, with the result

$$\int_{-\infty}^t dt' G_{1a} \bar{A}_{abc} v_{bm} v_{cn} = 2\eta^\perp v_{2m} v_{1n} + \int_{-\infty}^t dt' A_{ab} \frac{d}{dt'} [G_{1a} v_{bm} v_{1n}] + \int_{-\infty}^t dt' G_{1a} \bar{A}_{ab\bar{c}} v_{bm} v_{\bar{c}n}, \quad (3.2)$$

where the index \bar{c} does not take the value 1. Here we used that the linear solutions v_{am} are zero at $t = -\infty$ (by definition), that the Green's function $G_{1a}(t, t) = \delta_{1a}$, and that $A_{1b} = -2\eta^\perp \delta_{b2}$ (exact). We see that the first term on the right-hand side exactly cancels with the gauge correction (the second term in (3.1)) that is necessary to create a properly gauge-invariant second-order result.

We start by working out the second term on the right-hand side of (3.2). We find

$$I \equiv \int_{-\infty}^t dt' A_{ab} \frac{d}{dt'} [G_{1a} v_{bm} v_{1n}] = \gamma_*^2 \int_{-\infty}^t dt' A_{ab} \frac{d}{dt'} [G_{1a} \bar{v}_{bm} \bar{v}_{1n} \Theta(t' - t_*)] \\ = A_{ab*} G_{1a}(t, t_*) v_{bm*} v_{1n*} + \gamma_*^2 \Theta(t - t_*) \int_{t_*}^t dt' A_{ab} [G_{1d} A_{da} \bar{v}_{bm} \bar{v}_{1n} - G_{1a} A_{bd} \bar{v}_{dm} \bar{v}_{1n} \\ - G_{1a} \bar{v}_{bm} A_{1d} \bar{v}_{dn}] \\ = A_{ab*} G_{1a}(t, t_*) v_{bm*} v_{1n*} - \gamma_*^2 \Theta(t - t_*) \int_{t_*}^t dt' A_{ab} A_{1d} G_{1a} \bar{v}_{bm} \bar{v}_{dn}, \quad (3.3)$$

where, as before, a subscript $*$ means that a quantity is evaluated at t_* . Using the explicit form of the matrix \mathbf{A} (2.9) and the solutions v_{am} (2.28) this becomes

$$I = \gamma_*^2 \Theta(t - t_*) \delta_{m2} \delta_{n1} \left(-2\eta_*^\perp + \chi_* G_{12}(t, t_*) + A_{32*} G_{13}(t, t_*) - \chi_* A_{33*} G_{13}(t, t_*) \right) \\ + \gamma_*^2 \Theta(t - t_*) \delta_{m2} \delta_{n2} \int_{t_*}^t dt' 2\eta^\perp \bar{v}_{22} \left[-2\eta^\perp \bar{v}_{22} - G_{12} \bar{v}_{32} + A_{32} G_{13} \bar{v}_{22} + A_{33} G_{13} \bar{v}_{32} \right]. \quad (3.4)$$

From now on we will drop the overall step function, which just encodes the obvious condition that $t \geq t_*$. Realizing that $A_{32}\bar{v}_{22} + A_{33}\bar{v}_{32} = -\frac{d}{dt'}\bar{v}_{32}$ we can do an integration by parts:

$$\begin{aligned}
I = & \gamma_*^2 \delta_{m2} \left[\delta_{n1} \left(-2\eta_*^\perp + \chi_* G_{12}(t, t_*) + A_{32*} G_{13}(t, t_*) - \chi_* A_{33*} G_{13}(t, t_*) \right) \right. \\
& \left. - \delta_{n2} 2\eta_*^\perp \chi_* G_{13}(t, t_*) \right] \\
& + \gamma_*^2 \delta_{m2} \delta_{n2} \int_{t_*}^t dt' 2\eta^\perp \left[-2\eta^\perp (\bar{v}_{22})^2 + \left(-2G_{12} + A_{33} G_{13} + \frac{\dot{\eta}^\perp}{\eta^\perp} G_{13} \right) \bar{v}_{22} \bar{v}_{32} + G_{13} (\bar{v}_{32})^2 \right].
\end{aligned} \tag{3.5}$$

To this result we have to add the final term on the right-hand side of (3.2). Using the explicit expression for the matrix $\bar{\mathbf{A}}$ and doing some more integrations by parts this can be worked out further, as can be found in D.2. The final result for f_{NL} in the equal momenta limit is (including also the final term of (3.1))

$$-\frac{6}{5} f_{\text{NL}} = \frac{-2\bar{v}_{12}^2}{[1 + (\bar{v}_{12})^2]^2} \left(g_{\text{iso}} + g_{\text{sr}} + g_{\text{int}} \right), \tag{3.6}$$

where

$$\begin{aligned}
g_{\text{iso}} &= (\epsilon + \eta^\parallel) (\bar{v}_{22})^2 + \bar{v}_{22} \bar{v}_{32}, & g_{\text{sr}} &= -\frac{\epsilon_* + \eta_*^\parallel}{2\bar{v}_{12}^2} + \frac{\eta_*^\perp \bar{v}_{12}}{2} - \frac{3}{2} \left(\epsilon_* + \eta_*^\parallel - \chi_* + \frac{\eta_*^\perp}{\bar{v}_{12}} \right), \\
g_{\text{int}} &= -\int_{t_*}^t dt' \left[2(\eta^\perp)^2 (\bar{v}_{22})^2 + (\epsilon + \eta^\parallel) \bar{v}_{22} \bar{v}_{32} + (\bar{v}_{32})^2 - G_{13} \bar{v}_{22} (C \bar{v}_{22} + 9\eta^\perp \bar{v}_{32}) \right].
\end{aligned} \tag{3.7}$$

Here we have defined

$$C \equiv 12\eta^\perp \chi - 6\eta^\parallel \eta^\perp + 6(\eta^\parallel)^2 \eta^\perp + 6(\eta^\perp)^3 - 2\eta^\perp \xi^\parallel - 2\eta^\parallel \xi^\perp - \frac{3}{2} (\tilde{W}_{211} + \tilde{W}_{222}), \tag{3.8}$$

where as before $\tilde{W}_{lmn} \equiv (\sqrt{2\epsilon}/\kappa) W_{lmn}/(3H^2)$. We should add that although no slow-roll approximation has been used on super-horizon scales, we did assume slow roll to hold at horizon crossing, in order to use the analytic linear short-wavelength solutions (2.28) and to remove any dependence on the window function \mathcal{W} . Observations of the scalar spectral index seem to indicate that slow roll is a good approximation at horizon crossing. In a numerical treatment we could use the exact numerical solutions instead.

Looking at (3.6), which is one of the main results of this paper, we can draw a number of important conclusions. In the first place there is a part of f_{NL} , namely the first term in g_{sr} , that survives in the single-field limit. It corresponds to the single-field non-Gaussianity produced at horizon crossing and comes from the $b_{ia}^{(2)}$ source term. It agrees with the single-field result of Maldacena [5] for $f_{\text{NL}}^{(4)}$. The rest of the result is proportional to \bar{v}_{12} , which describes the contribution of the isocurvature mode to the adiabatic mode. In the single-field case it is identically zero, so that there is no super-horizon contribution to f_{NL} in that case. Moreover, since $\theta^1 = 2\eta^\perp \zeta^2$, such a contribution only builds up when η^\perp is non-zero, i.e. when the field trajectory makes a turn in field space. We also see that there are three different sorts of terms in the expression for f_{NL} . The g_{sr} terms are proportional to a slow-roll parameter evaluated at t_* and thus are always small because we assume slow roll to hold at

horizon-crossing. Although the terms proportional to \bar{v}_{12} and $1/\bar{v}_{12}$ in g_{sr} are time varying, one can easily show that neither $\bar{v}_{12}/(1 + \bar{v}_{12}^2)^2$ nor $\bar{v}_{12}^3/(1 + \bar{v}_{12}^2)^2$ are ever bigger than 0.33. The g_{iso} terms are proportional to \bar{v}_{22} , the pure isocurvature mode. These terms can be big, in particular during a turn in field space, but in the models that we consider, where the isocurvature mode has disappeared by the end of inflation, they become zero again and cannot lead to observable non-Gaussianities. The reason that we do not consider models with surviving isocurvature modes is that in that case the evolution after inflation is not clear. In the presence of isocurvature modes the adiabatic mode is not necessarily constant (indeed, that is the source of the non-Gaussianities we are considering here), which means that the final results at recombination might depend on the details of the transition at the end of inflation and of (p)reheating. Hence we will make sure that in all models we consider the isocurvature modes have disappeared by the end of inflation, which means in particular that the turn of the trajectory in field space has to occur a sufficient number of e-folds before the end of inflation. Note, however, that this is a constraint we impose voluntarily to simplify the evolution after inflation, it is in no way a necessary condition for the validity of our formalism during inflation. Finally, the third type of term in (3.6) is the integral in g_{int} . It is from this integrated effect that any large, persistent non-Gaussianity originates.

For completeness we also calculate the power spectrum, which according to equation (2.35) takes the simple form

$$\mathcal{P}_\zeta = \frac{\kappa^2 H_*^2}{8\pi^2 \epsilon_*} (1 + \bar{v}_{12}^2), \quad (3.9)$$

and the spectral index, calculated analytically using equations (2.36), (2.16) and (2.5),

$$n_\zeta - 1 = \frac{1}{1 - \epsilon_*} \left[-4\epsilon_* - 2\eta_*^\parallel + 2\frac{\bar{v}_{12}}{1 + \bar{v}_{12}^2} \left(-2\eta_*^\perp + \chi_* \bar{v}_{12} \right. \right. \\ \left. \left. + G_{13}(t, t_*) \left(-\tilde{W}_{221*} + 2\epsilon_*^2 + \eta_*^{\parallel 2} + \eta_*^{\perp 2} + 3\epsilon_*(\eta_*^\parallel - \chi_*) - 2\eta_*^\parallel \chi_* + \chi_*^2 \right) \right) \right] \quad (3.10)$$

3.2 General momenta

We turn now to the more general case where each scale exits the horizon at a different time t_{k_i} , defined by $aH = k_i c / \sqrt{2}$, where $c \approx 3$ is a constant allowing for some time to pass after horizon exit so that the long-wavelength approximation is valid (see the discussion in section 2.2). It is important to realize that it is not the momentum dependence of the bispectrum that we are discussing here, but of f_{NL} . The momentum dependence of the local bispectrum is dominated by the momentum dependence of the power spectrum squared, leading to the well-known result (see e.g. [24]) that it peaks on squeezed triangles where one of the momenta is much smaller than the other two. Here we are discussing the momentum dependence of f_{NL} , so one has divided by the power spectrum squared. This f_{NL} , often called $f_{NL}^{(4)}$ in the δN literature, is usually assumed to be momentum-independent. However, as we will show this is not true and its momentum dependence can lead to relative effects of order 10% even within the range of momenta that are observable by Planck.

Assuming $k_1 \geq k_2$, i.e. $t_{k_1} \geq t_{k_2}$, we find that (2.38) reduces to

$$\begin{aligned}
f(k_1, k_2) = & -\frac{\gamma_{k_1}^2 \gamma_{k_2}^2}{2} \bar{v}_{1mk_2}(t) \bar{v}_{1nk_1}(t) \left[\int_{t_{k_1}}^t dt' G_{1a}(t, t') [\bar{A}_{ab\bar{h}} + \bar{A}_{a\bar{h}b}] \bar{v}_{bm k_2}(t') \bar{v}_{\bar{h}n k_1}(t') \right. \\
& - \int_{t_{k_1}}^t dt' G_{1a}(t, t') [A_{ab} A_{1e} + A_{ae} A_{1b}] \bar{v}_{bm k_2}(t') \bar{v}_{en k_1}(t') \\
& + A_{ab}(t_{k_1}) G_{1a}(t, t_{k_1}) [\bar{v}_{bm k_2}(t_{k_1}) \delta_{1n} + \bar{v}_{1mk_2}(t_{k_1}) \delta_{bn}] \\
& \left. - G_{1a}(t, t_{k_1}) M_{abc}(t_{k_1}) [\bar{v}_{cm k_2}(t_{k_1}) \delta_{bn} + \bar{v}_{bm k_2}(t_{k_1}) \delta_{cn}] \right], \tag{3.11}
\end{aligned}$$

where again we have used the result (D.1) and have done an integration by parts that cancels the gauge correction term as in (3.2). The indices \bar{b} and \bar{h} do not take the value 1. We have introduced the notation $\bar{v}_{i1k_l} \equiv \delta_{i1}$ and $\bar{v}_{i2k_l} \equiv G_{i2}(t, t_{k_l}) - \chi_{k_l} G_{i3}(t, t_{k_l})$, where χ_{k_l} is evaluated at t_{k_l} . We notice that due to the step functions the integral's lower limit corresponds to the time when both scales have entered the long-wavelength regime, i.e. the time when the larger k_1 (smaller wavelength) exits the horizon. The expression has become more complicated as compared to (3.2) and (3.3) since the $\bar{v}_{bm k_i}$ refer to a different initial value depending on the horizon crossing time of each scale k_i . Following the same procedure as in the previous section we find that

$$-\frac{6}{5} f_{\text{NL}}(k_1, k_2, k_3) = \frac{f(k_1, k_2) + f(k_2, k_3) + f(k_1, k_3)}{\gamma_{k_1}^2 \gamma_{k_2}^2 [1 + (\bar{v}_{12k_1})^2][1 + (\bar{v}_{12k_2})^2] + 2 \text{ perms.}}, \tag{3.12}$$

where

$$f(k_1, k_2) = -2\gamma_{k_1}^2 \gamma_{k_2}^2 (\tilde{v}_{12})^2 \left(g_{iso}(k_1, k_2) + g_{sr}(k_1, k_2) + g_{int}(k_1, k_2) + g_k(k_1, k_2) \right) \tag{3.13}$$

with

$$\begin{aligned}
g_{iso}(k_1, k_2) &= (\epsilon + \eta^\parallel)(\tilde{v}_{22})^2 + \tilde{v}_{22} \tilde{v}_{32}, \\
g_{sr}(k_1, k_2) &= \eta_{k_1}^\perp \left(\frac{G_{22k_1 k_2} \bar{v}_{12k_1}}{2} - \frac{1}{\bar{v}_{12k_2}} - \frac{G_{22k_1 k_2}}{2\bar{v}_{12k_1}} \right) + \frac{3\chi_{k_2}}{4} G_{33k_1 k_2} \\
&\quad - \frac{3}{2}(\epsilon_{k_1} + \eta_{k_1}^\parallel) G_{22k_1 k_2} + \frac{\chi_{k_1}}{4} \left(2\frac{\bar{v}_{12k_1}}{\bar{v}_{12k_2}} + G_{22k_1 k_2} \right) - \frac{\epsilon_{k_1} + \eta_{k_1}^\parallel}{2(\tilde{v}_{12})^2}, \\
g_{int}(k_1, k_2) &= -\int_{t_{k_1}}^t dt' \left[2(\eta^\perp)^2 (\tilde{v}_{22})^2 + (\epsilon + \eta^\parallel) \tilde{v}_{22} \tilde{v}_{32} + (\tilde{v}_{32})^2 - G_{13} \tilde{v}_{22} (C \tilde{v}_{22} + 9\eta^\perp \tilde{v}_{32}) \right], \\
g_k(k_1, k_2) &= \frac{1}{4\bar{v}_{12k_1}} \left[3G_{13}(\chi_{k_1} G_{22k_1 k_2} - \chi_{k_2} G_{33k_1 k_2}) + G_{32k_1 k_2} \left((3 + \epsilon_{k_1} + 2\eta_{k_1}^\parallel) G_{13} - \bar{v}_{12k_1} \right) \right] \\
&\quad + \frac{1}{4} G_{12k_1 k_2} (-2\eta_{k_1}^\perp + \chi_{k_1} \bar{v}_{12k_1}) + \frac{1}{2} G_{32k_1 k_2} (\eta_{k_1}^\perp G_{13} - 1) \\
&\quad - \frac{G_{12k_1 k_2}}{2\bar{v}_{12k_1}} \left(\epsilon_{k_1} + \eta_{k_1}^\parallel + \eta_{k_1}^\perp \bar{v}_{12k_1} \right) - \frac{1}{2} \left(\epsilon_{k_1} + \eta_{k_1}^\parallel - \frac{\chi_{k_1}}{2} \right) G_{12k_1 k_2}, \tag{3.14}
\end{aligned}$$

for $k_1 \geq k_2 \geq k_3$ and C was defined in (3.8). We introduced the notation

$$\begin{aligned}
(\tilde{v}_{12})^2 &\equiv \bar{v}_{12k_1} \bar{v}_{12k_2}, & (\tilde{v}_{22})^2 &\equiv \bar{v}_{22k_1} \bar{v}_{22k_2}, & (\tilde{v}_{32})^2 &\equiv \bar{v}_{32k_1} \bar{v}_{32k_2}, \\
\tilde{v}_{22} \tilde{v}_{32} &\equiv \frac{1}{2} (\bar{v}_{22k_1} \bar{v}_{32k_2} + \bar{v}_{22k_2} \bar{v}_{32k_1}),
\end{aligned}$$

and also $G_{ijk_1k_2} \equiv G_{ij}(t_{k_1}, t_{k_2})$, while the subscript on the slow-roll parameters denotes evaluation at the relevant time that the scale exits the horizon. The Green's functions that appear without arguments denote $G(t, t_{k_1})$ outside or $G(t, t')$ inside the integral.

Although this expression is quite a bit longer than (3.6), there are many similarities between the two results. The whole expression is again proportional to \bar{v}_{12k_i} , except for the single-field horizon-crossing result, so that there is no super-horizon contribution to f_{NL} for the single-field case. In the g_{iso} and g_{sr} terms we recognize the familiar terms of the equal-momenta case, i.e. the isocurvature contributions proportional to \bar{v}_{22k_i} as well as the horizon crossing terms now evaluated at t_{k_1} and t_{k_2} (note that for $k_1 = k_2$, $G_{iik_1k_2} = 1$ identically and we regain the expressions of (3.7)). The integral has also retained its form. The rest of the terms, namely those in g_k , are terms arising due to the different horizon-crossing times of the scales and are identically zero for the equal-momenta case $k_1 = k_2$ where $G_{ijk_1k_2} = \delta_{ij}$. All terms inside g_k are proportional to a slow-roll parameter evaluated at horizon crossing (using the fact that $G_{13} = G_{12}/3$ up to slow-roll corrections, see (4.2)), except for the very last term on the second line. However, $G_{32k_1k_2}$ is expected to be quite small: for $k_1 = k_2$ it is zero, and for $k_1 \gg k_2$ it becomes the linear solution for the isocurvature velocity θ^2 (see (2.28)). Hence we do not expect g_k to give a large contribution, which is confirmed numerically. As we will see later on, the dominant contribution to the differences between different momentum configurations comes from the changes in the other terms.

4 Slow-roll approximation

While the exact result for f_{NL} , equation (3.6) or (3.12), is an extremely useful starting point for an exact numerical treatment, the integral cannot be done analytically. In order to find explicit analytic results that will be very useful to gain insight and draw generic conclusions, we need to simplify the problem by making the slow-roll approximation. In subsection 4.1 we further work out (3.6) under this approximation. Even then the integral can only be done analytically for certain specific classes of inflationary potentials, which are treated in the other subsections.

4.1 General expressions

Considering the slow-roll version of equation (2.21) we find that $g(t)$ (as defined above equation (2.23)) satisfies

$$\dot{g} + \chi g = 0. \quad (4.1)$$

We see that $Y(t) \propto \exp(-3t)$ so that $f(t)$ is a rapidly decaying solution that can be neglected (see (2.23) for definitions). After the decaying mode has vanished the solutions for the Green's functions simplify to

$$\begin{aligned} G_{22}(t, t') &= \frac{g(t)}{g(t')}, & G_{12}(t, t') &= \frac{2}{g(t')} \int_{t'}^t d\bar{t} \, \eta^\perp(\bar{t}) g(\bar{t}), \\ G_{32}(t, t') &= -\chi(t) G_{22}(t, t'), & G_{x3}(t, t') &= \frac{1}{3} G_{x2}(t, t'). \end{aligned} \quad (4.2)$$

4.1.1 Equal momenta

Using the last two relations in (4.2) and dropping higher-order terms in slow roll, (3.7) reduces to

$$g_{iso} = (\epsilon + \eta^\parallel - \chi)(\bar{v}_{22})^2, \quad g_{sr} = -\frac{\epsilon_* + \eta_*^\parallel}{2\bar{v}_{12}^2} + \frac{\eta_*^\perp \bar{v}_{12}}{2} - \frac{3}{2} \left(\epsilon_* + \eta_*^\parallel - \chi_* + \frac{\eta_*^\perp}{\bar{v}_{12}} \right), \quad (4.3)$$

$$g_{int} = \int_{t_*}^t dt' (\bar{v}_{22})^2 \left[2\eta^\perp \left(-\eta^\perp + \frac{(\epsilon + \eta^\parallel - \chi)\chi}{2\eta^\perp} \right) + G_{12} \left(\eta^\perp \chi - 2\eta^\parallel \eta^\perp - \frac{1}{2}(\tilde{W}_{211} + \tilde{W}_{222}) \right) \right].$$

Inserting these terms into (3.6) we find an expression that can be considered the final expression for f_{NL} in the slow-roll approximation, and is the one that will be used in section 4.3. It also proves useful, however, to rewrite it in a different way using integration by parts.

We use the slow-roll version of relation (2.22), $2\eta^\perp = -\frac{d}{dt'} G_{12}(t, t') + \chi G_{12}(t, t')$, to do an integration by parts, leading to

$$g_{int} = \bar{v}_{12} \left(-\eta_*^\perp + \frac{(\epsilon_* + \eta_*^\parallel - \chi_*)\chi_*}{2\eta_*^\perp} \right) + \int_{t_*}^t dt' G_{12}(\bar{v}_{22})^2 \left[2\eta^\perp \chi - \frac{(\epsilon + \eta^\parallel - \chi)\chi^2}{2\eta^\perp} \right. \\ \left. - 2\eta^\parallel \eta^\perp - \frac{1}{2}(\tilde{W}_{211} + \tilde{W}_{222}) + \frac{d}{dt'} \left(-\eta^\perp + \frac{(\epsilon + \eta^\parallel - \chi)\chi}{2\eta^\perp} \right) \right]. \quad (4.4)$$

Using the slow-roll version of the relations (D.11),

$$\xi^\parallel = 3\epsilon\eta^\parallel + (\eta^\parallel)^2 + (\eta^\perp)^2 - \tilde{W}_{111} \quad \text{and} \quad \xi^\perp = 3\epsilon\eta^\perp + 2\eta^\parallel \eta^\perp - \eta^\perp \chi - \tilde{W}_{211}, \quad (4.5)$$

as well as the time derivatives of the slow-roll parameters in (2.5), we can derive that

$$\frac{d}{dt} \left(-\eta^\perp + \frac{(\epsilon + \eta^\parallel - \chi)\chi}{2\eta^\perp} \right) = \frac{1}{2\eta^\perp} \left[-\chi^3 + (\epsilon + \eta^\parallel)\chi^2 - 4(\epsilon\eta^\parallel + (\eta^\perp)^2)\chi \right. \\ \left. + 4(\epsilon^2\eta^\parallel + \epsilon(\eta^\parallel)^2 - \epsilon(\eta^\perp)^2 + \eta^\parallel(\eta^\perp)^2) - (\epsilon + \eta^\parallel - \chi)\tilde{W}_{111} \right. \\ \left. + \left(2\eta^\perp + \frac{(\epsilon + \eta^\parallel - \chi)\chi}{\eta^\perp} \right) \tilde{W}_{211} + (\epsilon + \eta^\parallel - 2\chi)\tilde{W}_{221} \right]. \quad (4.6)$$

Inserting this into expression (4.4) for g_{int} and including the remaining terms in the expression for f_{NL} we finally obtain

$$-\frac{6}{5}f_{NL}(t) = \frac{-2(\bar{v}_{12})^2}{[1 + (\bar{v}_{12})^2]^2} \left\{ (\epsilon + \eta^\parallel - \chi)(\bar{v}_{22})^2 - \frac{\epsilon_* + \eta_*^\parallel}{2\bar{v}_{12}^2} + \frac{\eta_*^\perp \bar{v}_{12}}{2} + \frac{(\epsilon_* + \eta_*^\parallel - \chi_*)\chi_*}{2\eta_*^\perp} \bar{v}_{12} \right. \\ \left. - \frac{3}{2} \left(\epsilon_* + \eta_*^\parallel - \chi_* + \frac{\eta_*^\perp}{\bar{v}_{12}} \right) \right. \\ \left. + \int_{t_*}^t dt' G_{12}(\bar{v}_{22})^2 \left[2\frac{\epsilon\eta^\parallel}{\eta^\perp} \left(-\chi + \epsilon + \eta^\parallel - \frac{(\eta^\perp)^2}{\eta^\parallel} \right) + \frac{1}{2}(\tilde{W}_{211} - \tilde{W}_{222} - \frac{\chi}{\eta^\perp} \tilde{W}_{221}) \right. \right. \\ \left. \left. - \frac{\epsilon + \eta^\parallel - \chi}{2\eta^\perp} \left(\tilde{W}_{111} - \tilde{W}_{221} - \frac{\chi}{\eta^\perp} \tilde{W}_{211} \right) \right] \right\}. \quad (4.7)$$

This is the alternative final result for f_{NL} in the slow-roll approximation.

Equation (4.7), as well as (4.3), is characterized by the same features as the result of the exact formalism. We can easily distinguish the pure isocurvature \bar{v}_{22} term, which we assume to vanish before the end of inflation in order for the adiabatic mode to be constant after inflation, as well as the terms evaluated at the time of horizon crossing, which are expected to be small. Any remaining non-Gaussianity at recombination has to originate from the integral. In subsections 4.2 and 4.3 we will further work out the expressions of this section for the case of certain classes of potentials to gain insight into their non-Gaussian properties. But first we look at the momentum dependence of f_{NL} in section 4.1.2.

4.1.2 Squeezed limit

In this section we will calculate the slow-roll expression for f_{NL} in the case where $k \equiv k_3 \ll k_1 = k_2 \equiv k'$, what is usually referred to as the squeezed limit. Note that we assume $k_1 = k_2$ for simplicity, to keep the expressions manageable, it is not a necessary condition. We start with equation (3.12) and follow the procedure of the previous subsection, that is we use the slow-roll approximations (4.2) for the Green's functions and drop higher-order terms in slow roll. Since there are only two relevant scales the expression simplifies to give

$$-\frac{6}{5}f_{\text{NL}} = \frac{-2\bar{v}_{12k'}/[1 + (\bar{v}_{12k'})^2]}{\gamma^2[1 + (\bar{v}_{12k'})^2] + 2[1 + (\bar{v}_{12k})^2]} \left[\gamma^2 \bar{v}_{12k'} \left(g_{iso}(k', k') + g_{sr}(k', k') + g_{int}(k', k') \right) + 2\bar{v}_{12k} \left(g_{iso}(k', k) + g_{sr}(k', k) + g_{int}(k', k) + g_k(k', k) \right) \right], \quad (4.8)$$

where $\gamma \equiv \gamma_{k'}/\gamma_k$ and

$$\begin{aligned} g_{iso}(k', k) &= (\epsilon + \eta^\parallel - \chi) \bar{v}_{22k} \bar{v}_{22k'}, \\ g_{sr}(k', k) &= \eta_{k'}^\perp \left(\frac{G_{22k'k} \bar{v}_{12k'}}{2} - \frac{1}{\bar{v}_{12k}} - \frac{G_{22k'k}}{2\bar{v}_{12k'}} \right) + \frac{3\chi_k}{4} G_{33k'k} \\ &\quad - \frac{3}{2}(\epsilon_{k'} + \eta_{k'}^\parallel) G_{22k'k} + \frac{\chi_{k'}}{4} \left(2 \frac{\bar{v}_{12k'}}{\bar{v}_{12k}} + G_{22k'k} \right) - \frac{\epsilon_{k'} + \eta_{k'}^\parallel}{2\bar{v}_{12k} \bar{v}_{12k'}}, \\ g_{int}(k', k) &= \int_{t_{k'}}^t dt' \bar{v}_{22k} \bar{v}_{22k'} \left[2\eta^\perp \left(-\eta^\perp + \frac{(\epsilon + \eta^\parallel - \chi)\chi}{2\eta^\perp} \right) + G_{12} \left(\eta^\perp \chi - 2\eta^\parallel \eta^\perp - \frac{1}{2}(\tilde{W}_{211} + \tilde{W}_{222}) \right) \right], \\ g_k(k', k) &= \frac{1}{4}(\chi_{k'} G_{22k'k} - \chi_k G_{33k'k}) + \frac{1}{12} G_{32k'k} (-6 + \epsilon_{k'} + 2\eta_{k'}^\parallel + 2\eta_{k'}^\perp \bar{v}_{12k'}) \\ &\quad + G_{12k'k} \left(-\eta_{k'}^\perp - \frac{\epsilon_{k'} + \eta_{k'}^\parallel}{2} \left(1 + \frac{1}{\bar{v}_{12k'}} \right) + \frac{\chi_{k'}}{4} (1 + \bar{v}_{12k'}) \right). \end{aligned} \quad (4.9)$$

The first line of (4.8), proportional to γ^2 , comes from the $f(k', k')$ term and it is identical to expression (4.3). The difference is that now it occurs with a weight γ^2 compared to the terms originating from $f(k', k)$ that come with a weight 2. Obviously, in the case of equal momenta where $G_{ij k_1 k_2} = \delta_{ij}$, the expression reduces to equation (4.3).

The γ terms can be safely neglected in the squeezed limit because γ^2 scales as $e^{-3\Delta t}$, where Δt is the number of e-folds between horizon exit of the two scales. If for example the two scales exit the horizon with a delay $\Delta t \sim 7$, which corresponds to $k' \sim 1000k$, approximately the resolution of the Planck satellite, we find that $\gamma^2 \sim 10^{-9}$.

The functions \bar{v}_{12} and \bar{v}_{22} increase and decrease respectively (from their initial values 0 and 1) only a little until the turning of the fields. The later the relevant scale exits the horizon, the less time there is available for \bar{v}_{12} to evolve, so the smaller is the value of \bar{v}_{12} (and the larger for \bar{v}_{22}) during this period. During the turning of the fields isocurvature effects turn on. Both \bar{v}_{12} and \bar{v}_{22} vary wildly during this period. \bar{v}_{12} grows and reaches a constant value afterwards, while \bar{v}_{22} varies and reaches zero when isocurvature effects cease. In the models we studied we found that while during this period \bar{v}_{22} continues to behave in the same way, i.e. being larger for the scale that exits last, \bar{v}_{12} changes behaviour and also becomes larger for the scale that exits last. In the end we observe that f_{NL} in the squeezed limit is smaller than in the equal-momenta case that was treated in the previous subsection. The effect is particularly pronounced during the turn of the field trajectory, mainly due to g_{iso} . As we will show in section 5.1, these effects can reduce the value of f_{NL} during the turn of the field trajectory by 10% on scales that are within the resolution of Planck.

4.2 Potentials with equal powers

4.2.1 Quadratic potential

The quadratic potential has been widely examined in the past and it is known that it cannot produce large non-Gaussianity (see for example [19]). Here we use our results to analytically explain why. While the quadratic potential is a special case of the more general sum potential treated later on, it is still interesting to discuss it separately in a different way. We start by deriving the result that for a quadratic two-field potential within slow roll,

$$\chi = \frac{d}{dt} \ln \frac{\epsilon \eta^\perp}{\eta^\parallel}. \quad (4.10)$$

Working out the right-hand side, using (2.5), we find

$$\chi = 2\epsilon + \eta^\parallel - \frac{(\eta^\perp)^2}{\eta^\parallel} - \frac{\xi^\parallel}{\eta^\parallel} + \frac{\xi^\perp}{\eta^\perp}. \quad (4.11)$$

Inserting the relations (4.5) (with the third derivatives of the potential equal to zero, since we have a quadratic potential) this reduces to

$$\chi = \epsilon + \eta^\parallel - \frac{(\eta^\perp)^2}{\eta^\parallel}. \quad (4.12)$$

It can be checked that this result does indeed satisfy the general equation for the time derivative of χ (2.5) within the approximations made, and the remaining integration constant is fixed by realizing that this result has the proper limit in the single-field case. This concludes the proof of (4.10).

Since the third-order potential derivatives as well as the first term of the integral in (4.7) are identically zero, we find that for a quadratic potential the integral completely vanishes in the slow-roll approximation and no persistent large non-Gaussianity is produced. Numerically we find that even for large mass ratios, when during the turn of the field trajectory slow roll is broken, the integral is still approximately zero, see section 5.1.

Using this result (4.10) for χ we can also solve (4.1):

$$g(t) = \frac{\eta^\parallel}{\epsilon \eta^\perp}, \quad (4.13)$$

and hence find that

$$G_{22}(t, t') = \frac{\epsilon(t')\eta^\perp(t')}{\eta^\parallel(t')} \frac{\eta^\parallel(t)}{\epsilon(t)\eta^\perp(t)}, \quad G_{12}(t, t') = -\frac{\epsilon(t')\eta^\perp(t')}{\eta^\parallel(t')} \left(\frac{1}{\epsilon(t)} + 2t - \frac{1}{\epsilon(t')} - 2t' \right). \quad (4.14)$$

Note that even though $g(t)$ is a large quantity, of order inverse slow roll, it is still slowly varying, as we have shown, with its time derivative an order of slow roll smaller.

4.2.2 Potentials of the form $W = \alpha\phi^p + \beta\sigma^q$

For a potential of the form

$$W(\phi, \sigma) = \alpha\phi^p + \beta\sigma^q \quad (4.15)$$

we can work out explicitly the form of the integrand in equation (4.7). We have to use the slow-roll version of equations (2.3) and (2.4) to easily find after substitution that

$$g_{int} = -\int_{t_*}^t \frac{\alpha\beta p^4(y-1)\phi^{p-3}\sigma^{py-3} (y(py-1)\phi^2 + (p-1)\sigma^2) (\alpha^2\phi^{2p}\sigma^2 + \beta^2y^2\phi^2\sigma^{2py})^2}{2\kappa^4 (\alpha\phi^p + \beta\sigma^{py})^4 (\alpha(p-1)\phi^p\sigma^2 - \beta y(py-1)\phi^2\sigma^{py})^2} dt', \quad (4.16)$$

where $y \equiv q/p$.

From this expression we can derive an important result: for $y = 1$, i.e. $p = q$, we immediately see that the integral is zero. This means that no persistent non-Gaussianity can be produced after horizon exit for potentials of the form $W(\phi, \sigma) = \alpha\phi^p + \beta\sigma^p$, at least within the slow-roll approximation. This generalizes the result for the two-field quadratic potential of the previous subsection to any potential with two equal powers.

4.3 Other integrable forms of potentials

In general, the first step of finding an analytical expression for the integral g_{int} is to solve the differential equation (4.1) for g in order to determine the Green's functions. To do that, one tries to express χ as a time derivative of some other quantity. In the slow-roll limit

$$\tilde{W}_{11} = \epsilon - \eta^\parallel, \quad \tilde{W}_{21} = -\eta^\perp, \quad (4.17)$$

so that χ can be written as

$$\chi = 2\epsilon + \tilde{W}_{22} - \tilde{W}_{11}. \quad (4.18)$$

which, as can be checked, cannot be expressed as a derivative of a known quantity for a general potential. Thus we are forced to examine special classes of potentials.

4.3.1 Product potentials

First we consider potentials of the form

$$W(\phi, \sigma) = U(\phi)V(\sigma), \quad (4.19)$$

inspired by the analytical study done in [62, 63]. From our point of view, the advantage of these potentials is that their mixed second derivative $\tilde{W}_{\phi\sigma}$ can be expressed in terms of the first derivatives to finally give for the second-order derivatives of the potential in the adiabatic and isocurvature directions:

$$\begin{aligned} \tilde{W}_{11} &= \tilde{W}_{\phi\phi}e_{1\phi}^2 + \tilde{W}_{\sigma\sigma}e_{1\sigma}^2 + 4\epsilon e_{1\phi}^2 e_{1\sigma}^2, & \tilde{W}_{22} &= \tilde{W}_{\phi\phi}e_{1\sigma}^2 + \tilde{W}_{\sigma\sigma}e_{1\phi}^2 - 4\epsilon e_{1\phi}^2 e_{1\sigma}^2, \\ \tilde{W}_{21} &= \left(\tilde{W}_{\phi\phi} - \tilde{W}_{\sigma\sigma} + 2\epsilon(e_{1\sigma}^2 - e_{1\phi}^2) \right) e_{1\phi}e_{1\sigma}, \end{aligned} \quad (4.20)$$

where we used (A.8) to eliminate the unit vector \mathbf{e}_2 in terms of \mathbf{e}_1 . It is straightforward to show that the second-order derivatives in the directions of the basis vectors are related:

$$\frac{2\epsilon + \tilde{W}_{22} - \tilde{W}_{11}}{\tilde{W}_{21}} = \frac{e_{1\sigma}}{e_{1\phi}} - \frac{e_{1\phi}}{e_{1\sigma}}, \quad (4.21)$$

so that only two of them are independent. Now we can use (2.4) and the above results to write χ as

$$\chi = \tilde{W}_{21} \left(\frac{e_{1\sigma}}{e_{1\phi}} - \frac{e_{1\phi}}{e_{1\sigma}} \right) = -\frac{d}{dt} \ln(e_{1\phi}e_{1\sigma}), \quad (4.22)$$

where the derivatives of the unit vectors are given in (2.6). Hence looking at equation (4.1) we can identify the Green's function g to be

$$g(t) = e_{1\phi}(t)e_{1\sigma}(t). \quad (4.23)$$

After a few more manipulations the integrand of $G_{12}(t, t')$ in (4.2) takes the form

$$\eta^\perp(t)g(t) = \frac{1}{4} \frac{dS}{dt}, \quad (4.24)$$

where $S \equiv e_{1\phi}^2 - e_{1\sigma}^2$, so that the analytical form of the two independent linear perturbation solutions in the slow-roll approximation is ⁴

$$\bar{v}_{12} = \frac{S - S_*}{2e_{1\phi*}e_{1\sigma*}}, \quad \bar{v}_{22} = \frac{e_{1\phi}e_{1\sigma}}{e_{1\phi*}e_{1\sigma*}}. \quad (4.25)$$

The final step is to write the integrand of g_{int} in (4.3) in terms of the potential's derivatives and rearrange terms to form time derivatives. One can prove that then the integrand can be rewritten as

$$g_{int} = \frac{1}{1 - S_*^2} \int_{t_*}^t \frac{d}{dt'} \left[\left(S(t) - S(t') \right) \left(\tilde{W}_{\sigma\sigma}(t')e_{1\phi}^2(t') - \tilde{W}_{\phi\phi}(t')e_{1\sigma}^2(t') \right) \right] dt'. \quad (4.26)$$

After performing the integration and adding the rest of the terms we find that

$$-\frac{6}{5}f_{NL} = \frac{2(S - S_*)^2(S_*^2 - 1)}{(1 + S^2 - 2SS_*)^2} (g_{iso} + g_{sr} + g_{int}), \quad (4.27)$$

where now

$$\begin{aligned} g_{iso} &= \frac{S^2 - 1}{S_*^2 - 1} \left(\epsilon + \eta^\parallel - \chi \right), \\ g_{sr} &= -\frac{1}{2(S - S_*)} \left[\left(\epsilon_* + \eta_*^\parallel \right) \frac{1 + 3S(S - 2S_*) + 2S_*^2}{S - S_*} - \chi_* \frac{-3 + S^2 + 4SS_* - 2S_*^2}{2S_*} \right], \\ g_{int} &= -\frac{S_*(S - S_*)}{S_*^2 - 1} \left(\epsilon_* + \eta_*^\parallel - \chi_* \frac{S_*^2 + 1}{2S_*^2} \right). \end{aligned} \quad (4.28)$$

Comparing to the results of [62, 63] we find complete agreement.

⁴Note added: Very recently the same results were obtained in [74] for the transfer functions $T_{\mathcal{RS}}$ and T_{SS} of product and sum potentials, which turn out to coincide with \bar{v}_{12} and \bar{v}_{22} .

Looking at the result for f_{NL} for the product potential we can draw a number of conclusions. The only time-dependent slow-roll parameters appear in g_{iso} . These terms and consequently f_{NL} can vary significantly during a turn of the field trajectory but, as explained before, in the models we consider isocurvature modes have disappeared by the end of inflation so that the adiabatic mode will be constant after inflation, which means g_{iso} will disappear again and cannot give any persistent non-Gaussianity. The rest of the terms involve slow-roll parameters evaluated at horizon crossing, which are small. Hence we conclude that any large non-Gaussianity will have to come from the denominator becoming very small (since $|S| \leq 1$ the numerator cannot become large) to compensate for the small slow-roll parameters at horizon crossing. We see that this can only happen when $S, S_* \rightarrow \pm 1$. In the remainder of this section we will study the two different cases that satisfy this condition: a 90° turn in the field trajectory ($S = -S_*$), or the same field dominating both at the beginning and at the end ($S = S_*$).

First we study the case where the field trajectory makes a 90° turn. The field ϕ is dominant right after horizon crossing, which means $|e_{1\sigma*}| \ll 1$, $|e_{1\phi*}| \approx 1$ and hence $S_* \rightarrow 1$. Later on occurs a turn in the field trajectory and afterwards σ leads inflation, so that $|e_{1\phi}| \ll 1$, $|e_{1\sigma}| \approx 1$ and $S \rightarrow -1$. Then we find that both g_{sr} and g_{iso} go to zero, which means in particular that we satisfy the condition on the disappearance of the isocurvature mode that allows us to directly extrapolate the results at the end of inflation to the time of recombination. The non-zero term comes as expected from g_{int} and it is given by:

$$-\frac{6}{5}f_{\text{NL}} = \epsilon_* + \eta_*^\parallel - \chi_* = -\tilde{W}_{\sigma\sigma*}, \quad (4.29)$$

since $\tilde{W}_{\sigma\sigma*} = \tilde{W}_{22*} = \chi_* - \epsilon_* - \eta_*^\parallel$. Hence we see that for any product potential where the field trajectory makes a 90° turn no significant non-Gaussianity will be produced, at least within the slow-roll assumptions used to derive this analytic result.

Next we look at the opposite limit, where one of the fields, ϕ , is dominant both at horizon crossing and at the end of inflation. This means $|e_{1\sigma*}| \ll 1$, $|e_{1\phi*}| \approx 1$ and $|e_{1\sigma}| \ll 1$, $|e_{1\phi}| \approx 1$, so that $S_* \rightarrow 1$ and $S \rightarrow 1$. This includes the case where we have a perfectly straight field trajectory, i.e. an effectively single-field situation, where obviously no super-horizon non-Gaussianity is produced. However, we find that even more generally in this limit the contributions from g_{iso} and g_{int} go to zero and we are left with only the single-field result from g_{sr} :

$$-\frac{6}{5}f_{\text{NL}} = \epsilon_* + \eta_*^\parallel. \quad (4.30)$$

Hence no significant non-Gaussianity is produced in this limit.

We conclude that if we impose the condition of the disappearance of the isocurvature mode by the end of inflation, to simplify the evolution afterwards, the product potential can never give large non-Gaussianity, at least within the slow-roll approximation.

4.3.2 Potentials of the form $W(\phi, \sigma) = (U(\phi) + V(\sigma))^\nu$

Next we consider potentials of the form

$$W(\phi, \sigma) = (U(\phi) + V(\sigma))^\nu, \quad (4.31)$$

which, to our knowledge, have not been worked out before for general ν .⁵ While of course not the most general two-field potential, it can accommodate potentials with coupling terms of the form $\alpha^2\phi^2 + \beta^2\sigma^2 + 2\alpha\beta\phi\sigma$ or higher-order combinations. Note that in the case of $\nu = 1$ the potential becomes the simple sum potential, which has been studied before [19, 63].

Just as for the product potential, we find that mixed second derivatives of the potential can be expressed in terms of the other derivatives:

$$\begin{aligned}\tilde{W}_{11} &= \tilde{W}_{\phi\phi}e_{1\phi}^2 + \tilde{W}_{\sigma\sigma}e_{1\sigma}^2 + \frac{4\epsilon(\nu-1)e_{1\phi}^2e_{1\sigma}^2}{\nu}, \quad \tilde{W}_{22} = \tilde{W}_{\sigma\sigma}e_{1\phi}^2 + \tilde{W}_{\phi\phi}e_{1\sigma}^2 - \frac{4\epsilon(\nu-1)e_{1\phi}^2e_{1\sigma}^2}{\nu}, \\ \tilde{W}_{21} &= \left(\tilde{W}_{\phi\phi} - \tilde{W}_{\sigma\sigma}\right)e_{1\phi}e_{1\sigma} + \frac{2\epsilon(\nu-1)e_{1\phi}e_{1\sigma}(e_{1\sigma}^2 - e_{1\phi}^2)}{\nu}.\end{aligned}\quad (4.32)$$

Again there are only two independent second derivatives of the potential in our basis:

$$\tilde{W}_{21} = \left(\tilde{W}_{22} - \tilde{W}_{11} + \frac{2\epsilon(\nu-1)}{\nu}\right) \frac{e_{1\sigma}e_{1\phi}}{e_{1\sigma}^2 - e_{1\phi}^2}. \quad (4.33)$$

Following the procedure of the previous section we rewrite χ as

$$\chi = \frac{2\epsilon}{\nu} + \tilde{W}_{21} \left(\frac{e_{1\sigma}}{e_{1\phi}} - \frac{e_{1\phi}}{e_{1\sigma}} \right) = -\frac{d}{dt} \ln \left(H^{2/\nu} e_{1\phi} e_{1\sigma} \right), \quad (4.34)$$

and then find an analytical expression for g ,

$$g(t) = H^{2/\nu}(t) e_{1\phi}(t) e_{1\sigma}(t). \quad (4.35)$$

The integrand of $G_{12}(t, t')$ is now written as

$$\eta^\perp(t)g(t) = \frac{1}{2} \left(\frac{\kappa^2}{3} \right)^{1/\nu} \frac{dZ}{dt}, \quad (4.36)$$

where $Z \equiv V e_{1\phi}^2 - U e_{1\sigma}^2$. Finally we find that

$$\bar{v}_{12} = \frac{Z - Z_*}{W_*^{1/\nu} e_{1\phi*} e_{1\sigma*}}, \quad \bar{v}_{22} = \frac{W^{1/\nu} e_{1\phi} e_{1\sigma}}{W_*^{1/\nu} e_{1\phi*} e_{1\sigma*}}. \quad (4.37)$$

Rewriting the integrand of g_{int} in terms of the potential's derivatives yields after a few manipulations

$$\begin{aligned}g_{int} &= \frac{W_*^{-2/\nu}}{e_{1\phi*}^2 e_{1\sigma*}^2} \int_{t_*}^t dt' \left\{ \frac{d}{dt'} \left[\frac{W^{2/\nu}(t') \epsilon(t') e_{1\phi}^2(t') e_{1\sigma}^2(t')}{\nu} \right] \right. \\ &\quad \left. + \frac{d}{dt'} \left[W^{1/\nu}(t') \frac{Z(t) - Z(t')}{2} \left(\tilde{W}_{\sigma\sigma}(t') e_{1\phi}^2(t') - \tilde{W}_{\phi\phi}(t') e_{1\sigma}^2(t') \right) \right] \right\} \quad (4.38)\end{aligned}$$

and adding the rest of the terms results in

$$-\frac{6}{5}f_{\text{NL}} = -\frac{2W_*^{2/\nu}(Z - Z_*)^2 e_{1\phi*}^2 e_{1\sigma*}^2}{\left(e_{1\sigma*}^2(Z + U_*)^2 + e_{1\phi*}^2(Z - V_*)^2 \right)^2} \left(g_{iso} + g_{sr} + g_{int} \right), \quad (4.39)$$

⁵Note added: While we were doing the final editing of our manuscript, a paper [75] appeared on the arXiv where the authors studied this type of potential using the δN formalism. Their result for f_{NL} agrees with ours.

where

$$\begin{aligned}
g_{iso} &= \left(\frac{W^{1/\nu} e_{1\phi} e_{1\sigma}}{W_*^{1/\nu} e_{1\phi*} e_{1\sigma*}} \right)^2 \left(\epsilon + \eta^\parallel - \chi \right), \\
g_{sr} &= -\frac{3}{2}(\epsilon_* + \eta_*^\parallel - \chi_*) + \frac{Z - Z_*}{W_*^{1/\nu} (e_{1\phi*}^2 - e_{1\sigma*}^2)} \left(-\frac{\epsilon_*}{\nu} + \frac{\chi_*}{2} \right) \left[1 - \frac{3W_*^{2/\nu} e_{1\phi*}^2 e_{1\sigma*}^2}{(Z - Z_*)^2} \right] \\
&\quad - (\epsilon_* + \eta_*^\parallel) \frac{W_*^{2/\nu} e_{1\phi*}^2 e_{1\sigma*}^2}{2(Z - Z_*)^2} \\
g_{int} &= \frac{Z - Z_*}{2W_*^{1/\nu}} \left(\frac{1}{e_{1\sigma*}^2} - \frac{1}{e_{1\phi*}^2} \right) \left(\epsilon_* + \eta_*^\parallel - \frac{\chi_*}{2} \left(1 + \frac{1}{(e_{1\phi*}^2 - e_{1\sigma*}^2)^2} \right) \right) \\
&\quad + \frac{\epsilon}{\nu} \left(\frac{W^{1/\nu} e_{1\phi} e_{1\sigma}}{W_*^{1/\nu} e_{1\phi*} e_{1\sigma*}} \right)^2 - \frac{\epsilon_*}{\nu} \left(1 - \frac{2(Z - Z_*)}{W_*^{1/\nu} (e_{1\phi*}^2 - e_{1\sigma*}^2)} \right). \tag{4.40}
\end{aligned}$$

Note that the first term on the second line of g_{int} is also related to the pure isocurvature mode, but we have not incorporated it in g_{iso} in order to remind the reader that it originates from the integral.

As in the case of the product potential we will study two limiting cases, to get some insight into the behaviour of f_{NL} . First is the limit where the field trajectory makes a 90° turn. We assume that ϕ dominates inflation at horizon exit, that is $|e_{1\sigma*}| \ll 1$, $|e_{1\phi*}| \approx 1$ and $Z_* \rightarrow V_*$. At late times, after the turn of the field trajectory, the second field σ is dominant and the contribution of ϕ is negligible, so that $|e_{1\phi}| \ll 1$, $|e_{1\sigma}| \approx 1$ and $Z \rightarrow -U$. Then we find that g_{sr} and g_{iso} go to zero, while the remaining contribution to f_{NL} comes from g_{int} , as expected,

$$-\frac{6}{5}f_{NL} = -\frac{U_* + V_*}{U + V_*} \tilde{W}_{\sigma\sigma*} = \frac{U_* + V_*}{U + V_*} (\epsilon_* + \eta_*^\parallel - \chi_*). \tag{4.41}$$

So we see that we need a significant decrease in U between horizon crossing and the end of inflation, as well as a relatively small value of V_* , to get a large f_{NL} . Of course we cannot increase U_*/U too much without breaking slow roll. In section 5.2 we investigate numerically the properties of a model with a sum potential and confirm the validity of the above limit.

In the opposite limit ϕ dominates both at horizon crossing and at the end of inflation, i.e. $|e_{1\sigma*}| \ll 1$, $|e_{1\phi*}| \approx 1$ and $|e_{1\sigma}| \ll 1$, $|e_{1\phi}| \approx 1$ so that $Z_* \rightarrow V_*$ and $Z \rightarrow V$. Then the expression reduces to

$$-\frac{6}{5}f_{NL} = -\frac{U_* + V_*}{V - V_*} (\epsilon_* + \eta_*^\parallel - \chi_*), \tag{4.42}$$

which comes from g_{int} . Note that we have assumed here that $V \neq V_*$. In the (effectively) single-field case this is not valid; in that case we find that g_{int} and g_{iso} are zero and g_{sr} goes to the single-field result, $\epsilon_* + \eta_*^\parallel$. We remark that in this limit g_{iso} is zero, so that the adiabatic mode is conserved after inflation. In order to make f_{NL} large, one might be tempted to take V close to V_* . However, that means σ does not evolve and we are in an effectively single-field situation, where the above limit is not valid. Instead the situation is somewhat similar to the previous limit: we need a large value of U_* and relatively small values of V_* and V to overcome the small values of the slow-roll parameters at horizon crossing. It might not be simple to satisfy these conditions together with the requirements of this limiting case that ϕ

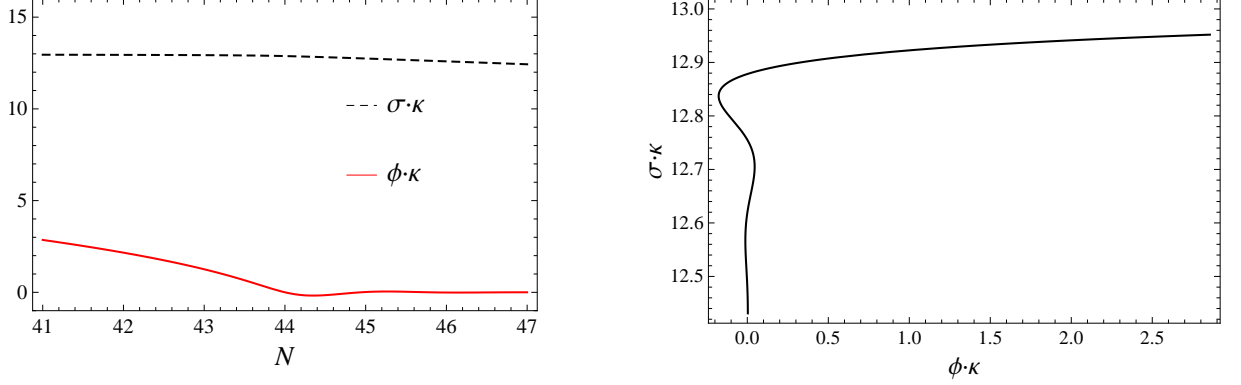


Figure 1. The time evolution of the fields (left) and the field trajectory (right) during the period of the turn of the field trajectory, for the model (5.1) with initial conditions $\phi_0 = \sigma_0 = 13\kappa^{-1}$ and mass ratio $m_\phi/m_\sigma = 20$.

dominates both at horizon crossing and at the end of inflation, with a period of σ domination in between; we did not further study those types of models.

As a final remark we point out that the power ν of the potential does not appear explicitly in the limits for f_{NL} . Of course its value will play a role in determining the field trajectory and the values of the slow-roll parameters, but that is only a relatively small effect. We have verified this result numerically for several values of the power ν of sum potentials of the form (4.15).

5 Numerical results

The formalism we have developed so far provides a tool to calculate the exact amount of non-Gaussianity produced during inflation driven by a general two-field potential, beyond the slow-roll approximation. While we assumed slow roll in the previous section, in order to derive analytical results, we return here to the exact formalism for a numerical treatment. In the following subsections we investigate the properties of the quadratic potential as well as a potential of the sum type that can produce an f_{NL} of the order of a few, and compare our results to those of the δN -formalism.

5.1 Comparison with δN for the quadratic potential

We investigate the quadratic potential

$$W = \frac{1}{2}m_\phi^2\phi^2 + \frac{1}{2}m_\sigma^2\sigma^2 \quad (5.1)$$

choosing our parameters as follows: $m_\phi/m_\sigma = 20$, $m_\sigma = 10^{-5}\kappa^{-1}$ and the initial conditions $\phi_0 = \sigma_0 = 13\kappa^{-1}$ at $t = 0$ for a total of about 85 e-folds of inflation. From now on we will denote the heavy field as ϕ . We choose to present this particular mass ratio because the fields oscillate wildly during the turn and slow roll is badly broken, so that it provides a serious check both of our formalism and the δN one. Of course we have also run tests with smaller mass ratios when slow roll is unbroken and verified our analytical slow-roll results.

We solve the field equations (2.3) numerically and in figures 1 and 2 we plot the values of the fields, the unit vectors, and the slow-roll parameters as a function of time during

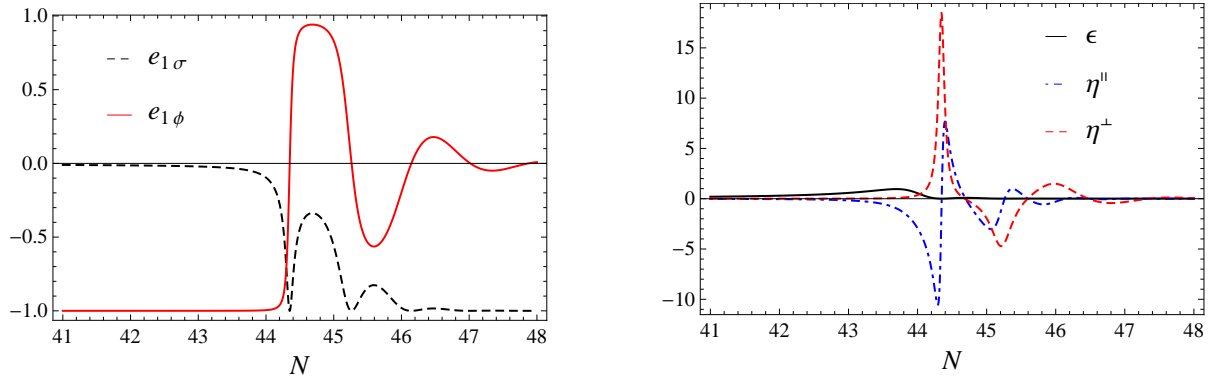


Figure 2. The unit vectors (left) and the slow-roll parameters $\epsilon, \eta^{\parallel}$ and η^{\perp} (right) as a function of time during the turn of the field trajectory, for the same model as in figure 1.

the range of e-folds where the heavy field ϕ is approaching zero and starts oscillating. In the beginning of inflation ϕ dominates the expansion while rolling down its potential and about 40 e-folds after the initial time $t = 0$ it starts oscillating around the minimum of its potential. The heavier ϕ is, the more persistent are the damped oscillations. During the period of oscillations the unit vectors, as well as the slow-roll parameters $\epsilon, \eta^{\parallel}$, and η^{\perp} , oscillate too. For $m_{\phi}/m_{\sigma} = 20$ the maxima of the slow-roll parameters are much larger than unity and slow roll is temporarily broken. During these oscillations the light field σ starts driving inflation and rolls down its potential until it also reaches its minimum and starts oscillating. We take the end of inflation when $\epsilon = 1$ during this second period of oscillations. The situation is similar to the limiting case we studied in section 4.3.2 with $|e_{1\sigma*}| \ll 1$ and $|e_{1\phi}| \ll 1$.

In figure 3 we plot the f_{NL} parameter as calculated in our formalism both the numerical exact version (3.6) and the slow-roll analytical approximation (4.7) (but using the exact background), as well as the result computed numerically in the context of the δN -formalism. The horizon-crossing time is defined as 60 e-folds before the end of inflation. We do not expect any large non-Gaussianity to be produced in this model, since we have shown that the integral of (4.7) is equal to zero in the slow-roll approximation. The final value of f_{NL} calculated in all three cases is $\mathcal{O}(10^{-2})$. Our results coincide completely with those of the δN -formalism, thus reinforcing the validity of both formalisms. We also show f_{NL} for a much smaller mass ratio, $m_{\phi}/m_{\sigma} = 4$, where slow roll remains valid throughout the turn of the field trajectory, verifying our analytical slow-roll result.

The peak of the f_{NL} parameter during the turning of the fields is due to the isocurvature terms g_{iso} in the slow-roll analytical formula. As expected this effect is transient and disappears when the isocurvature mode \bar{v}_{22} has been fully converted to the adiabatic one. There is no surviving isocurvature mode in this model. The higher is the mass ratio, the larger is the magnitude of the peak as a consequence of the more violent oscillations.

For completeness, we plot in figure 4 the power spectrum (3.9) and the spectral index (3.10) of this model. We see there is a jump in both of them during the oscillatory period of the heavy field, but afterwards they become constant again.

Finally in figure 5 we plot the exact numerical f_{NL} in the squeezed limit and in the equal momenta limit. As mentioned in section 4.1.2, we see that the f_{NL} parameter in the squeezed limit is smaller than in the equal-momenta one. From figure 5 we can see that for

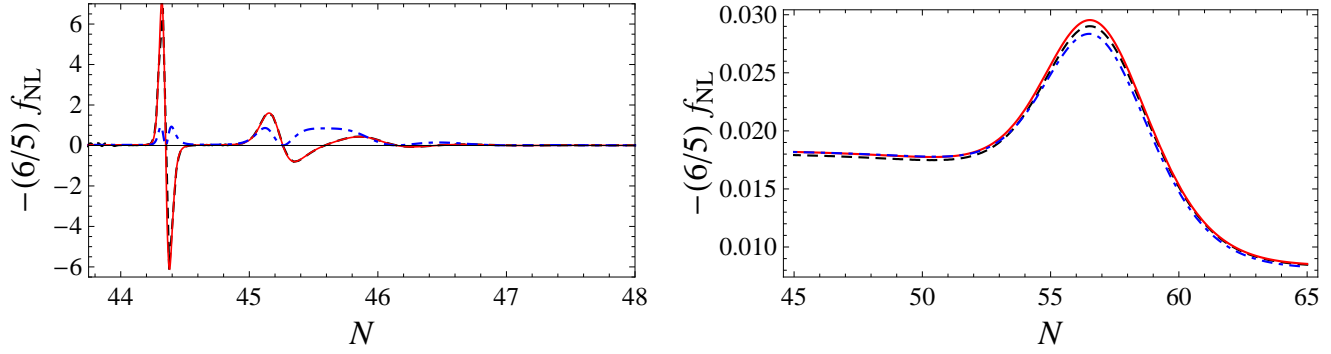


Figure 3. We plot the f_{NL} parameter for the model (5.1) with initial conditions $\phi_0 = \sigma_0 = 13\kappa^{-1}$ and mass ratio $m_\phi/m_\sigma = 20$ (left) and $m_\phi/m_\sigma = 4$ (right). The red line is the exact numerical result, while the blue dot-dashed line shows the slow-roll analytical approximation (but using the exact background). We also show the numerical δN result as the black dashed line, which lies practically on top of our red result.

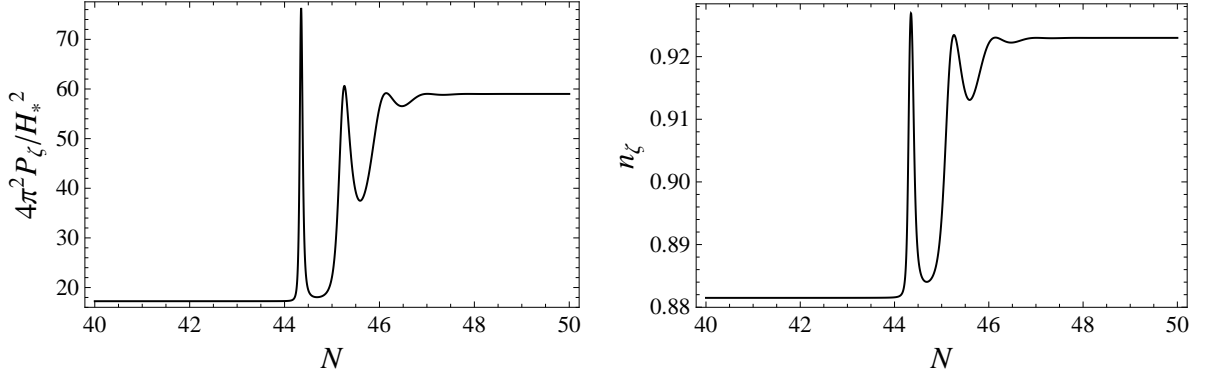


Figure 4. The exact numerical power spectrum (left) and the spectral index (right) for the same model as in figure 1.

$k' = 1000k$ (roughly corresponding to the Planck resolution) the peak value of f_{NL} is more than 10% smaller than for $k' = k$, for this particular model.

5.2 A simple model producing large non-Gaussianity

In this section we introduce a model that produces an f_{NL} of the order of a few, which is two orders of magnitude larger than the single-field slow-roll result. So in that sense we can call it large. From the point of view of observations with the Planck satellite it is probably still a little bit too small, but we have taken this particular model to be able to make the connection with our analytical results.

The f_{NL} limit (4.41) that we calculated in section 4.3.2 can be simplified for the sum potential ($\nu = 1$) to give

$$-\frac{6}{5}f_{\text{NL}} = -\frac{V_{\sigma\sigma*}}{\kappa^2(U + V_*)}, \quad (5.2)$$

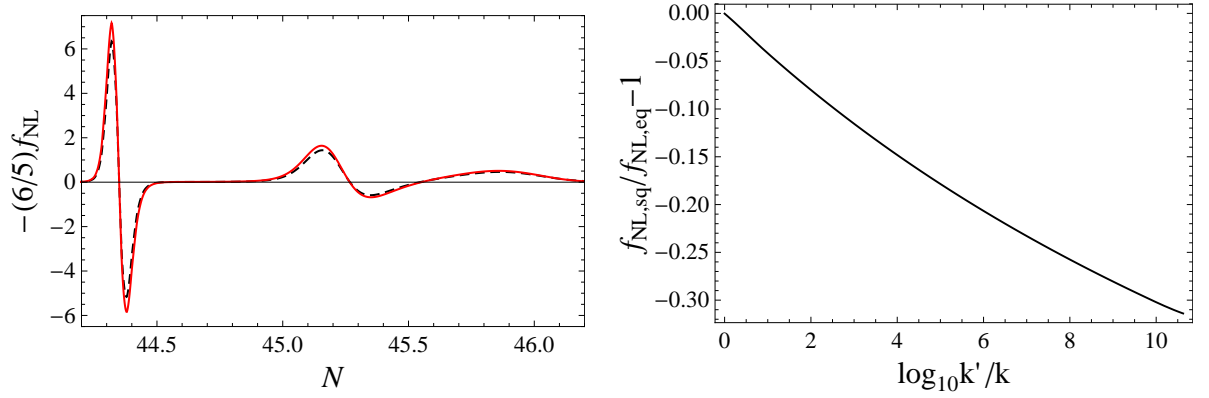


Figure 5. In the first plot we depict the f_{NL} parameter for the equal momenta limit in red and the squeezed limit result for $k'/k = 1000$ in dashed black. In the second plot we show the dependence of the discrepancy of the first peak value of f_{NL} in the two limits on the ratio k'/k . We used the same model as in figure 1 and $t_{k'} = 25$ (60 e-folds before the end of inflation).

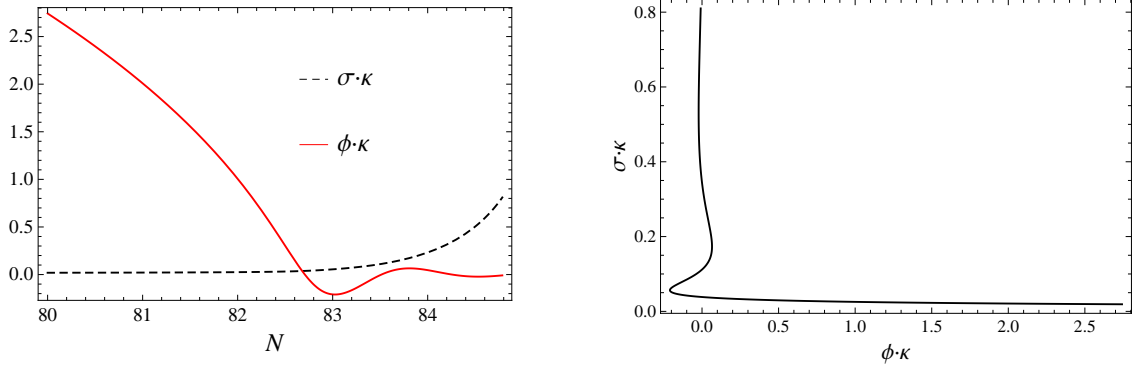


Figure 6. The time evolution of the fields (left) and the field trajectory (right), for the model (5.3) with initial conditions $\phi_0 = 18\kappa^{-1}$, $\sigma_0 = 0.01\kappa^{-1}$ and parameters $a_2 = 20\kappa^{-2}$, $b_2 = 7\kappa^{-2}$ and $b_4 = 2$. Only the time interval during the turn of the field trajectory is shown.

where we used the definition of \tilde{W}_{mn} and the slow-roll version of (2.3) for H . We can easily infer that in order to obtain a large value for f_{NL} , the heavy field ϕ should end up with a small value at the end of inflation, while σ should obey a potential characterized by a large second derivative and a small value at horizon crossing. Such properties can be accommodated by a potential of the form

$$\begin{aligned} U(\phi) &= a_2\phi^2, \\ V(\sigma) &= b_0 - b_2\sigma^2 + b_4\sigma^4, \end{aligned} \quad (5.3)$$

with $b_0 = b_2^2/(4b_4)$ so that the minimum of the potential has $W = U + V = 0$.

To illustrate the above we investigate a model with $a_2 = 20\kappa^{-2}$, $b_2 = 7\kappa^{-2}$, $b_4 = 2$, and initial conditions $\phi_0 = 18\kappa^{-1}$ and $\sigma_0 = 0.01\kappa^{-1}$, so that the light field is standing on the local maximum of its potential, for a total amount of 85 e-folds of inflation. This type of effective potential might be realized in the early universe during second-order phase transitions. We

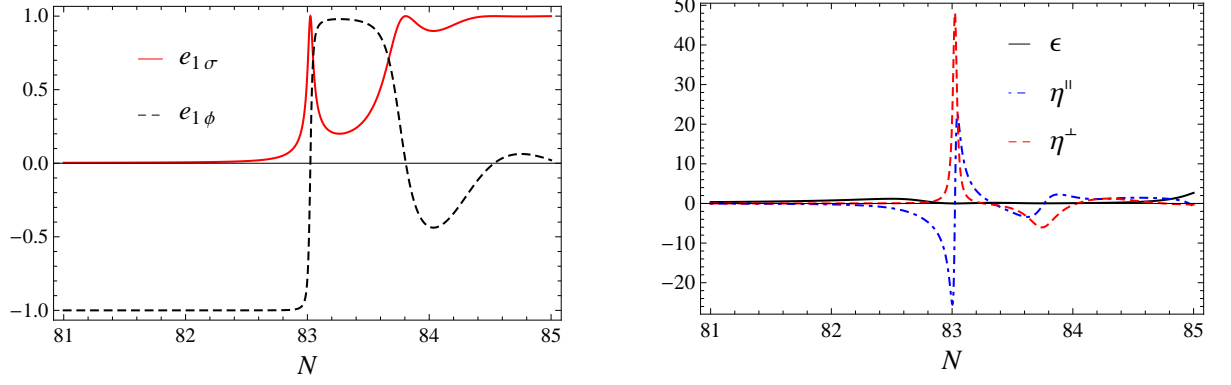


Figure 7. The unit vectors (left) and the slow-roll parameters $\epsilon, \eta^{\parallel}$ and η^{\perp} (right) as a function of time during the turn of the field trajectory, for the same model as in figure 6.

solve the field equations (2.3) numerically and in figures 6 and 7 we plot the evolution of the fields and the unit vectors, as well as the slow-roll parameters. The situation is qualitatively the same as in the case of the quadratic potential: in the beginning ϕ dominates inflation while rolling down its potential, then there is a period of violent oscillations around $\phi = 0$, and σ takes over and starts rolling down towards the minimum of its potential.

The behaviour of the unit vectors is that of the limiting case we studied in section 4.3.2, that is $|e_{1\sigma}| \ll 1$ and $|e_{1\phi}| \ll 1$. We will try to obtain an analytical estimate for the magnitude of f_{NL} . The final value of f_{NL} is reached when the fields have rolled down to their minima, that is at $\phi = 0$ and $\sigma = \sqrt{b_2/(2b_4)}$ for a positive initial condition for σ . Then f_{NL} becomes

$$-\frac{6}{5}f_{\text{NL}} = 8 \frac{\omega(1 - 6\omega\sigma_*^2)}{\kappa^2(1 - 2\omega\sigma_*^2)^2}, \quad (5.4)$$

where $\omega \equiv b_4/b_2$. Note that within our approximation f_{NL} depends only on the value of σ_* at horizon crossing once we have fixed the ratio ω . Since the turning of the fields occurs only a few e-folds before the end of inflation, we will explicitly assume $W \simeq U$ is a good approximation for nearly all the period of inflation. Then we can solve the field equations in the slow-roll approximation to find

$$\phi(t) = \phi_0 \sqrt{1 - \frac{4t}{\kappa^2 \phi_0^2}}, \quad \sigma(t) = \left[2\omega - \left(2\omega - \frac{1}{\sigma_0^2} \right) \left(1 - \frac{4t}{\kappa^2 \phi_0^2} \right)^r \right]^{-1/2}, \quad (5.5)$$

where $r \equiv b_2/a_2$.

The time of horizon crossing $t_* = t_{\text{fin}} - 60$ can be approximately found from the final time $t_{\text{fin}} \simeq \kappa^2 \phi_0^2/4$ and thus we calculate the values of the fields at horizon exit as functions of the initial conditions ϕ_0 and σ_0 . Using these results in f_{NL} we find

$$-\frac{6}{5}f_{\text{NL}} = -\frac{8\omega}{\kappa^2(1 - 2\omega\sigma_0^2)^2} \left[2 \left(-1 + \tilde{\phi}_0^{2r} \right) \omega \sigma_0^2 + 1 \right] \left[2 \left(1 + 2\tilde{\phi}_0^{2r} \right) \omega \sigma_0^2 - 1 \right], \quad (5.6)$$

where $\tilde{\phi}_0 = \phi_0/(2\sqrt{60}/\kappa) = \phi_0/\phi_*$.

We now check the dependence of the above expression on the initial condition σ_0 . Since we assumed that $|e_{1\sigma*}| \ll 1$ and $W \simeq U$ we examine the case $\sigma_0 \ll 1$ where

$$-\frac{6}{5}f_{\text{NL}} = \frac{8\omega}{\kappa^2}(1 - 2\omega\sigma_0^2\tilde{\phi}_0^{2r}) \quad (5.7)$$

up to second order with respect to σ_0 . The parameter f_{NL} becomes maximal if $\omega = \tilde{\phi}_0^{-2r}/(4\sigma_0^2)$ and its value is then

$$-\frac{6}{5}f_{\text{NL}} = \frac{\tilde{\phi}_0^{-2r}}{\kappa^2\sigma_0^2}. \quad (5.8)$$

Since $\tilde{\phi}_0 > 1$, the smaller the ratio r and the smaller the initial value of the field σ , the higher is the value of f_{NL} .

Nevertheless one has to assure that the turn of the field trajectory does not occur too late (too close to the end of inflation), so that the isocurvature mode will have had the time to disappear before the end of inflation (so that we can directly extrapolate the results at the end of inflation to the time of recombination and do not have to take further evolutionary effects into account) and the oscillations of the heavy field do not coincide with those of the light field. The higher is the ratio ω , the larger is f_{NL} , but then the minimum of the potential approaches σ_0 and consequently there is less time available for \bar{v}_{12} and thus for the adiabatic perturbation to become constant. This turns out to be a non-trivial requirement: although we do not claim to have scanned the whole parameter space of the model, we could not find parameter values that passed the above test and at the same time yielded a very large f_{NL} . The values we have chosen to work with in this paper respect the above condition and using expression (5.7) we expect to find $-(6/5)f_{\text{NL}} \sim 2$.

If one were to take $b_4 = 5$ instead of 2, one would find $-(6/5)f_{\text{NL}} \sim 4$, but in that case the turn of the fields occurs too near the end of inflation so that the isocurvature mode will not have disappeared completely by the end of inflation. Looking at the contributions of g_{iso} and g_{int} separately, we see that even in that case g_{int} has already gone to a constant while g_{iso} is still decreasing towards zero, so that we feel reasonably confident that the estimate is good even for that model, but we cannot be absolutely certain without a better treatment of the end of inflation, which is beyond the scope of the present paper.

In figure 8 we plot f_{NL} for the model (5.3) with the parameter values described above. Again a notable feature comes up during the turn of the fields. It comes from the isocurvature term of (4.39) that gets very big during the turn of the field trajectory, but as soon as the fields relax it vanishes again. We do not plot g_{sr} separately since it turns out to be negligible. Note how the final value of f_{NL} depends only on the integrated effect, as the isocurvature contribution has vanished. The final slow-roll analytical value is calculated to be $-(6/5)f_{\text{NL},sr} = 2.15$ while the values obtained numerically by our formalism and the δN formalism are $-(6/5)f_{\text{NL}} = 1.43$ and $-(6/5)f_{\text{NL},\delta N} = 1.48$, respectively. We see excellent agreement between the exact numerical result of our formalism and the δN one. The slow-roll analytical result does very badly during the turn of the field trajectory, when slow roll is badly broken, but gives a reasonable estimate (within 50%) of the final value.

Finally in figure 9 we plot the spectral index for this model. Its value is in the range of the 68% confidence levels after 7 years of WMAP observations [1], but lies near the upper limit.

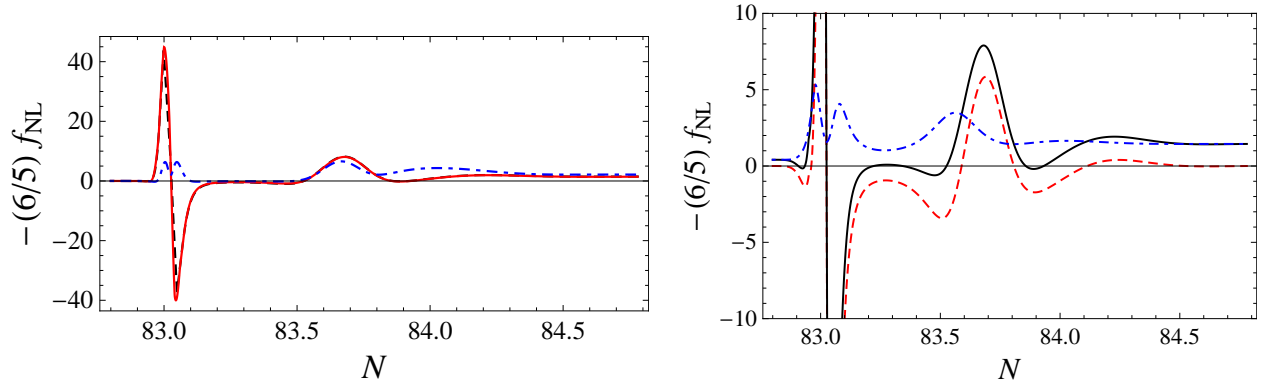


Figure 8. In the first plot we show the non-Gaussianity parameter f_{NL} as calculated in our formalism exactly (red line) and within the analytical slow-roll approximation (blue dot-dashed line) as well as numerically in the context of the δN formalism (black dashed line) for the model (5.3). In the second plot we show again the total f_{NL} , now as the black solid line, and split it up into the isocurvature contribution proportional to g_{iso} (red dashed line) and the integral contribution proportional to g_{int} (blue dot-dashed line). We use the same model as in figure 6.

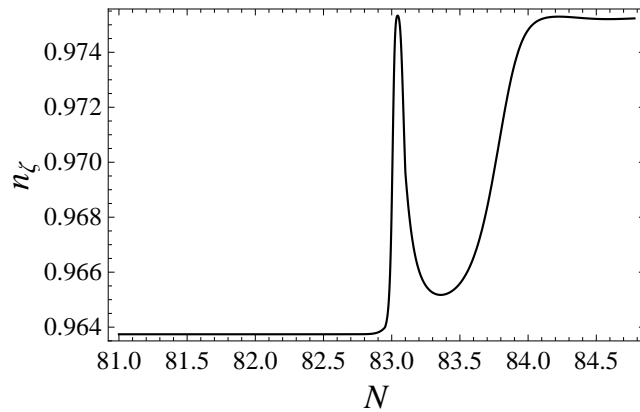


Figure 9. The exact numerical spectral index for the same model as in figure 6.

6 Conclusions

The study of the non-Gaussianity produced by inflation models has become a hot topic of research, since the recent observations of WMAP and in particular the imminent ones of Planck will allow us to constrain and discriminate inflation models based on their non-Gaussian predictions. In this paper we investigated the super-horizon bispectral non-Gaussianity produced by two-field inflation models. To this end we further worked out the long-wavelength formalism developed by Rigopoulos, Shellard, and Van Tent (RSvT) [6, 12, 13, 76].

We derived an exact result for the bispectrum parameter f_{NL} produced on super-horizon scales for any two-field inflation model with canonical kinetic terms, equation (3.6). The result is expressed in terms of the linear perturbation solutions and slow-roll parameters. However, no slow-roll approximation has been assumed on super-horizon scales, these parameters should be viewed as short-hand notation and can be large. In particular this means that the

result is valid for models where the field trajectory makes a sharp turn in field space so that slow roll is temporarily broken. On the other hand, we did assume slow roll to be valid at horizon crossing in order to remove any dependence on the window function and to use the analytic solutions for the linear mode functions. Observations of the scalar spectral index seem to indicate that this is a good approximation. Note that the assumption of canonical kinetic terms is not a fundamental one: the basic equations of the formalism of RSvT are given for more general kinetic terms. We just did not want to complicate the notation and expressions in this paper with the covariant derivatives and additional curvature terms needed to treat the general case.

The result can be split into the sum of three parts, multiplied by an overall factor (except for a small slow-roll suppressed term that is the single-field contribution produced at horizon crossing). This overall factor is proportional to the contribution of the isocurvature mode to the adiabatic mode, which is only non-zero for a truly multiple-field model where the field trajectory makes a turn in field space, as parametrized by a non-zero value of the slow-roll parameter η^\perp . (Effectively) single-field models do not produce any non-Gaussianity on super-horizon scales, since the adiabatic perturbation is conserved in that case. The three parts in the sum are: 1) a part that only involves slow-roll parameters evaluated at horizon-crossing and hence is always small; 2) a part proportional to the pure isocurvature mode; and 3) an integral involving terms proportional to the pure isocurvature mode. Since the adiabatic mode is not necessarily constant in the presence of isocurvature modes, we only consider models where the isocurvature mode has disappeared by the end of inflation, so that we can directly extrapolate our result at the end of inflation to recombination and observations of the CMB. However, this automatically means that the part 2), although varying wildly during the turn of the field trajectory, cannot give any persistent non-Gaussianity that can be observed in the CMB. This means that any large non-Gaussianity on super-horizon scales in models satisfying this condition will have to come from the integrated effect in part 3).

The exact equation (3.6) is the basis of our numerical studies. However, to gain further insight we tried to work out the integral analytically. For this it turns out that the slow-roll approximation is necessary. Even then the integral can only be done explicitly for certain specific classes of potentials, among which are product potentials, $W(\phi, \sigma) = U(\phi)V(\sigma)$, and generalized sum potentials, $W(\phi, \sigma) = (U(\phi) + V(\sigma))^\nu$. We found that, with our assumptions on the disappearance of the isocurvature mode, no product potential can give large non-Gaussianity, nor can any simple sum potential with equal powers, $W(\phi, \sigma) = \alpha\phi^p + \beta\sigma^p$. However, we found conditions under which the (generalized) sum potential can give large non-Gaussianity (here defined as f_{NL} larger than unity), and we have described an explicit, simple model that does. It consists of a heavy field rolling down a quadratic potential while a light field sits near the local maximum of a double-well potential. When the heavy field reaches zero and starts oscillating, the light field takes over and rolls down, so that there is a turn of the field trajectory in field space. We studied this model numerically, using the exact results, to confirm our analytical predictions.

In deriving equation (3.6) we assumed that all three scales cross the horizon at the same moment. However, this is not a necessary assumption, and we also generalized the result to an arbitrary momentum configuration. We find that going to the squeezed limit, where one of the momenta is much smaller than the other two, even when remaining within the resolution of the Planck satellite ($k' \sim 1000k$), the result for f_{NL} can be reduced by about 10%, depending on the model. We stress that we are discussing f_{NL} here, so this effect is unrelated to the well-known result that the local bispectrum peaks on squeezed momentum

configurations, which is due to the momentum behaviour of the power spectrum, which has been divided out in f_{NL} . However, exactly because of this latter effect, the squeezed limit is very relevant for the computation of f_{NL} .

We have worked out and included the second-order source term at horizon crossing in the long-wavelength formalism of RSvT, a contribution that had been missing so far. This is the only change of the basic formalism with respect to the paper [6] by RSvT. While this additional term is always small for the models we consider and hence numerically insignificant, from an analytical point of view it means we could now compare our results directly to the so-called $f_{\text{NL}}^{(4)}$ as defined in the δN formalism [19, 62, 63]. Some of the potentials we studied had already been worked out using that formalism and where available we compared our analytical results and found perfect agreement. We also compared our exact numerical results with those obtained using a numerical δN treatment for models where slow roll is broken and the analytic results cannot be trusted, and again we found excellent agreement.

We showed that the long-wavelength formalism of RSvT represents a viable alternative to the δN formalism to compute the super-horizon non-Gaussianity produced during inflation, allowing us to obtain and verify results in a different way. Moreover, the long-wavelength formalism has a number of advantages that can make it preferable in certain situations. Very importantly, our formalism allows for a simple physical interpretation of the different parts in terms of adiabatic and isocurvature modes, providing insight into the behaviour of the different transient and persistent contributions to f_{NL} . While we did not pursue this in the present paper, the formalism also provides the solution for the second-order isocurvature perturbation and hence the isocurvature bispectrum could be computed as easily as the adiabatic one.

From our studies it has become clear that the condition on the disappearance of the isocurvature mode by the end of inflation is a very strong constraint. It significantly reduces the possibilities for a large, observable value of f_{NL} produced during inflation. Note, however, that we chose to impose this condition only to be able to neglect the further evolution of the adiabatic mode after inflation; it is in no way a necessary condition for our formalism during inflation. In future work we would like to relax this condition, which means that the adiabatic mode would no longer necessarily be constant after inflation, and hence will require a much better description and understanding of the evolution of the perturbations during the transition at the end of inflation and the subsequent period of (p)reheating. In conclusion, while a lot of progress has been made over the past few years regarding the non-Gaussianity produced in multiple-field inflation, more work still remains to be done.

Acknowledgments

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A Basis improvements

In [8, 77] an orthonormal basis in field space was introduced, with substantial advantages for calculating and interpreting quantities in multiple-field inflation (see also [78] for a special two-field case of this basis). The basis was defined as follows (note that e.g. $\mathbf{\Pi}$ is the vector

containing components Π^A). The first basis vector \mathbf{e}_1 is the unit vector in the direction of the field velocity. Next, the direction of the basis vector \mathbf{e}_2 is given by the direction of that part of the field acceleration that is perpendicular to \mathbf{e}_1 . This orthogonalization process is then continued with higher-order time derivatives, until a complete basis is found. Defining the generalized $\boldsymbol{\eta}$ parameter as

$$\boldsymbol{\eta}^{(n)} \equiv \frac{(\frac{1}{N}\partial_t)^{n-1}\boldsymbol{\Pi}}{H^{n-1}\Pi} \quad (\text{A.1})$$

(with $\boldsymbol{\eta} \equiv \boldsymbol{\eta}^{(2)}$ and $\boldsymbol{\xi} \equiv \boldsymbol{\eta}^{(3)}$), we can then define the basis vectors via an iterative procedure as

$$\mathbf{e}_n \equiv \frac{\boldsymbol{\eta}^{(n)} - \sum_{i=1}^{n-1} \eta_i^{(n)} \mathbf{e}_i}{\eta_n^{(n)}} \quad (\text{A.2})$$

for $n \geq 2$, with $\mathbf{e}_1 \equiv \boldsymbol{\Pi}/\Pi$ and $\eta_i^{(n)} \equiv \mathbf{e}_i \cdot \boldsymbol{\eta}^{(n)}$. Basically there is an arbitrariness in the choice of sign of the basis vectors, which in the original definition was fixed by choosing $\eta_n^{(n)}$ to be non-negative:

$$\eta_n^{(n)} \equiv \left| \boldsymbol{\eta}^{(n)} - \sum_{i=1}^{n-1} \eta_i^{(n)} \mathbf{e}_i \right|. \quad (\text{old definition}) \quad (\text{A.3})$$

While being a perfectly valid choice analytically, this choice does mean that certain basis vector components and slow-roll parameters make sudden sign flips when one or more fields are oscillating, and that is hard to deal with numerically. Hence we now propose a different choice for $\eta_n^{(n)}$, which is identical except for the overall sign, and which eliminates the sudden sign flips:

$$\eta_n^{(n)} \equiv -\varepsilon_{A_1 \dots A_n} e_1^{A_1} \dots e_{n-1}^{A_{n-1}} \eta^{(n) A_n}, \quad (\text{new definition}) \quad (\text{A.4})$$

where ε is the fully antisymmetric symbol. From the fact that $\boldsymbol{\eta} = \sum_{i=1}^n \eta_i^{(n)} \mathbf{e}_i$ it immediately follows that

$$\varepsilon_{A_1 \dots A_n} e_1^{A_1} \dots e_n^{A_n} = -1, \quad (\text{A.5})$$

so that this choice means that the basis has a definite handedness. Note that in the case where the fields do not oscillate, the two definitions have the same overall sign (hence the choice of the minus sign). To have the expressions for the time derivative of the basis vectors and the $\eta_n^{(n)}$ unchanged, we see that we also need the relation

$$\varepsilon_{A_1 \dots A_n} e_1^{A_1} \dots e_{n-1}^{A_{n-1}} e_{n+1}^{A_n} = 0 \quad (\text{A.6})$$

to be satisfied. Then all results and expressions developed with this basis are unchanged.

An interesting consequence of these relations, including the orthogonality relation

$$\mathbf{e}_m \cdot \mathbf{e}_n = \delta_{mn}, \quad (\text{A.7})$$

is that for the cases of two and of three fields we have sufficient conditions to write all basis vectors in terms of \mathbf{e}_1 , without knowing anything about the dynamics. For two fields we have

$$\mathbf{e}_2 = (e_1^2, -e_1^1), \quad (\text{A.8})$$

with $(e_1^1)^2 + (e_1^2)^2 = 1$, and for three fields

$$\begin{aligned} \mathbf{e}_2 &= (e_1^2 + e_1^3, -e_1^1 + e_1^3, -e_1^1 - e_1^2), \\ \mathbf{e}_3 &= \left(\frac{1}{2} - (e_1^1)^2 - e_1^2 e_1^3, -\frac{1}{2} + (e_1^2)^2 - e_1^1 e_1^3, \frac{1}{2} - (e_1^3)^2 - e_1^1 e_1^2 \right), \end{aligned} \quad (\text{A.9})$$

with $(e_1^1)^2 + (e_1^2)^2 + (e_1^3)^2 = 1$ and $e_1^1 e_1^2 - e_1^1 e_1^3 + e_1^2 e_1^3 = -\frac{1}{2}$.

B Computation of the second-order source term

To compute the second-order source term $b_{ia}^{(2)}$ of (2.13) we choose a gauge characterized by the constraint $e_{1A}\varphi^A = 0$, so that $\zeta_1 = \alpha$ and $\zeta_2 = -(\kappa/\sqrt{2\epsilon})e_{2A}\varphi^A$, where we have split up $\phi^A(t, \mathbf{x}) = \phi^A(t) + \varphi^A(t, \mathbf{x})$ and $\ln a(t, \mathbf{x}) = \ln a(t) + \alpha(t, \mathbf{x})$. Note that we use here subscripts 1 and 2 to indicate the adiabatic and isocurvature components of ζ (instead of superscripts). On long wavelengths this gauge reduces to the uniform energy density gauge. We will drop the tilde (that we introduced in section 2.3 to denote quantities in the uniform energy density gauge) and omit the superscript ⁽¹⁾ on first-order quantities in this appendix in order to lighten the notation.

We can use the momentum constraint (the equivalent of (2.3) in the proper gauge) to solve for the lapse function N to first order by setting $N = 1 + N_1$, and we find $N_1 = \dot{\zeta}/H$. Following Maldacena [5] we first write the quadratic action in the ADM formalism. For that we only need to compute N_1 to first order. We find that to leading order in slow roll the action takes the form

$$S_2 = \int dt d^3x L_2 = \int dt d^3x a^3 \epsilon \left\{ \dot{\zeta}_1^2 + \dot{\zeta}_2^2 - 4\eta^\perp H \dot{\zeta}_1 \dot{\zeta}_2 + 2\chi H \dot{\zeta}_2 \dot{\zeta}_2 \right\}. \quad (\text{B.1})$$

To obtain this expression we used the gauge constraint to simplify the integrand as well as (2.4) and (2.5).

For the third-order action we find for the leading-order slow-roll terms [69]

$$S_3 = \int dt d^3x \left\{ a^3 \epsilon^2 (\text{interaction terms}) - \frac{\delta L_2}{\delta \zeta_2} \left(-\frac{Q_2}{2} + (\epsilon + \eta^\parallel) \zeta_1 \zeta_2 + \frac{\dot{\zeta}_2 \zeta_1}{H} + \frac{\eta^\perp}{2} \zeta_1^2 \right) - \frac{\delta L_2}{\delta \zeta_1} \left(-\frac{\zeta_1^{(2)}}{2} - \frac{Q_1}{2} + \frac{\epsilon + \eta^\parallel}{2} \zeta_1^2 - \eta^\perp \zeta_1 \zeta_2 + \frac{\dot{\zeta}_1 \zeta_1}{H} \right) \right\}, \quad (\text{B.2})$$

where the auxiliary quantities Q_i are defined as

$$Q_i = -\frac{H}{\phi} e_{iA} \varphi_{(2)}^A, \quad (\text{B.3})$$

and the last two terms of the action are proportional to the second-order equations of motion. In order for them to vanish we perform a redefinition of ζ_1 and ζ_2 :

$$\begin{aligned} \zeta_1 &= \zeta_{1c} - \frac{\zeta_1^{(2)}}{2} - \frac{Q_1}{2} + \frac{\epsilon + \eta^\parallel}{2} \zeta_1^2 - \eta^\perp \zeta_1 \zeta_2 + \frac{\dot{\zeta}_1 \zeta_1}{H}, \\ \zeta_2 &= \zeta_{2c} - \frac{Q_2}{2} + (\epsilon + \eta^\parallel) \zeta_1 \zeta_2 + \frac{\dot{\zeta}_2 \zeta_1}{H} + \frac{\eta^\perp}{2} \zeta_1^2. \end{aligned} \quad (\text{B.4})$$

The terms Q_i can be rewritten using the definitions of the gradients of the curvature perturbations at second order. While working in the uniform energy density gauge we can use the equivalent of (2.3) to find the following constraints to first and second order [69]:

$$\begin{aligned} H \dot{\phi}_A \partial_i \varphi_{(1)}^A &= 0, \\ \frac{1}{2} Q_1 &= \frac{\epsilon + \eta^\parallel}{2} \zeta_2^2 + \frac{\dot{\zeta}_2 \zeta_2}{H} - \partial^{-2} \partial^i \left(\frac{\zeta_2}{H} \partial_i \dot{\zeta}_2 \right), \end{aligned} \quad (\text{B.5})$$

while expanding ζ_{2i} up to second order in the same gauge gives

$$\frac{1}{2}\zeta_{2i}^{(2)} = \frac{1}{2}\partial_i\zeta_2^{(2)} = \partial_i\left[\frac{1}{2}Q_2 + \frac{1}{H}\zeta_2\dot{\zeta}_1 - \frac{\eta^\perp}{2}\zeta_2^2 - \partial^{-2}\partial^j\left(\frac{\zeta_2}{H}\partial_j\dot{\zeta}_1\right)\right]. \quad (\text{B.6})$$

Then the redefinitions of the gradients become

$$\zeta_1 + \frac{\zeta_1^{(2)}}{2} = \zeta_{1c} + \frac{\dot{\zeta}_1\zeta_1}{H} - \frac{\dot{\zeta}_2\zeta_2}{H} + \frac{\epsilon + \eta^\parallel}{2}(\zeta_1^2 - \zeta_2^2) - \eta^\perp\zeta_1\zeta_2 + \partial^{-2}\partial^i\left(\frac{\zeta_2}{H}\partial_i\dot{\zeta}_2\right), \quad (\text{B.7})$$

$$\zeta_2 + \frac{\zeta_2^{(2)}}{2} = \zeta_{2c} + \frac{\zeta_2\dot{\zeta}_1}{H} + \frac{\zeta_1\dot{\zeta}_2}{H} + \frac{\eta^\perp}{2}(\zeta_1^2 - \zeta_2^2) + (\epsilon + \eta^\parallel)\zeta_1\zeta_2 - \partial^{-2}\partial^i\left(\frac{\zeta_2}{H}\partial_i\dot{\zeta}_1\right). \quad (\text{B.8})$$

Inspecting (2.30) and (B.6) we see that in the uniform energy density gauge $\zeta_{mi} = \partial_i\zeta_m$ both for the adiabatic and the isocurvature component. Since we perform our main calculation in the flat gauge and we use the variable ζ_{mi} rather than ζ_m , we want to transform the above redefinitions to this gauge by the simple gauge transformation (C.3):

$$\zeta_{mi}^{(2)} = \tilde{\zeta}_{mi}^{(2)} - \frac{\zeta_1}{H}\dot{\zeta}_{mi}, \quad (\text{B.9})$$

where we used again the tilde to denote the uniform energy density gauge. We find

$$\zeta_{1i} + \frac{\zeta_{1i}^{(2)}}{2} = \partial_i\left[\zeta_{1c} + \frac{\dot{\zeta}_1\zeta_1}{H} - \frac{\dot{\zeta}_2\zeta_2}{H} + \frac{\epsilon + \eta^\parallel}{2}(\zeta_1^2 - \zeta_2^2) - \eta^\perp\zeta_1\zeta_2 + \partial^{-2}\partial^j\left(\frac{\zeta_2}{H}\partial_j\dot{\zeta}_2\right)\right] - \frac{\zeta_1}{H}\partial_i\dot{\zeta}_1 \quad (\text{B.10})$$

$$\zeta_{2i} + \frac{\zeta_{2i}^{(2)}}{2} = \partial_i\left[\zeta_{2c} + \frac{\zeta_2\dot{\zeta}_1}{H} + \frac{\zeta_1\dot{\zeta}_2}{H} + \frac{\eta^\perp}{2}(\zeta_1^2 - \zeta_2^2) + (\epsilon + \eta^\parallel)\zeta_1\zeta_2 - \partial^{-2}\partial^j\left(\frac{\zeta_2}{H}\partial_j\dot{\zeta}_1\right)\right] - \frac{\zeta_1}{H}\partial_i\dot{\zeta}_2 \quad (\text{B.11})$$

Note that after horizon exit $\dot{\zeta}_1 = 2H\eta^\perp\zeta_2$ and $\dot{\zeta}_2 = -H\chi\zeta_2$ (the latter is valid under the slow-roll approximation only) in the gauge used in this appendix, so that the expressions simplify. For a field redefinition of the form $\zeta = \zeta_c + \lambda\zeta_c^2$ (note that in the equations above we have not added the subscript c explicitly in the quadratic terms, since to second order it makes no difference) the three-point correlation function can be written as

$$\langle\zeta\zeta\zeta\rangle = \langle\zeta_c\zeta_c\zeta_c\rangle + 2\lambda[\langle\zeta_c\zeta_c\rangle\langle\zeta_c\zeta_c\rangle + \text{cyclic}]. \quad (\text{B.12})$$

Hence the elements of L_{1ab} and N_{1ab} are just the coefficients of the various products of ζ_1 and ζ_2 in the redefinition of ζ_1 multiplied by 2, and similarly for L_{2ab} and N_{2ab} . Note that the local terms L_{abc} correspond to the terms between the square brackets, and the non-local terms N_{abc} to the terms outside. This leads to the explicit slow-roll expressions given at the end of section 2.2.

C Gauge issues

C.1 Gauge invariant quantities

The use of spatial gradients was first advocated in [79] in the context of the covariant formalism. Later on the authors of [12] constructed invariant quantities under long-wavelength

changes of time-slicing, by considering the following combination of the spatial gradients of two spacetime scalars A and B :

$$\mathcal{C}_i \equiv \partial_i A - \frac{\partial_t A}{\partial_t B} \partial_i B, \quad (\text{C.1})$$

and formed among others the quantity

$$\zeta_i = \partial_i \alpha - \frac{\partial_t \alpha}{\partial_t \rho} \partial_i \rho, \quad (\text{C.2})$$

which when linearized is just the gradient of the comoving curvature perturbation ζ . Here α is the logarithm of the scale factor and ρ the energy density. In [55] it was argued that this particular combination could give rise to a second-order gauge-invariant quantity $\zeta^{(2)}$ as follows:

$$\partial_i \zeta^{(2)} = \zeta_i^{(2)} - \frac{\rho^{(1)}}{\dot{\rho}} \dot{\zeta}_i^{(1)}, \quad (\text{C.3})$$

where $\zeta^{(2)}$ can be related to yet another gauge-invariant second-order quantity defined by Malik and Wands in [53],

$$\zeta^{(2)} \simeq \zeta_{MW}^{(2)} - \zeta_{MW}^{(1)2}. \quad (\text{C.4})$$

We can easily show that (C.3) is equivalent to taking the gauge transformation of $\zeta_i^{(2)1}$ from homogeneous expansion time slices to uniform energy density time slices. Denoting as tilded the quantities in the new time slices, where $T = t + \Delta t$ (no space transformation is required), we can write

$$\begin{aligned} \tilde{\zeta}_i^{(1)1} &= \zeta_i^{(1)1}, \\ \tilde{\zeta}_i^{(2)1} &= \zeta_i^{(2)1} - \Delta t \dot{\zeta}_i^{(1)1}. \end{aligned} \quad (\text{C.5})$$

The quantity Δt is the time difference between the two time slices and can be evaluated by either comparing α (see also (2.31)) or ρ in the two gauges:

$$\rho(t, x) = \tilde{\rho}(t + \Delta t) = \tilde{\rho}(t) + \Delta t \dot{\tilde{\rho}}(t), \quad (\text{C.6})$$

which expanded to first order becomes

$$\rho^{(0)}(t) + \rho^{(1)}(t, x) = \tilde{\rho}(t) + \Delta t \dot{\tilde{\rho}}(t). \quad (\text{C.7})$$

Hence we find that $\rho^{(0)} = \tilde{\rho}$ and

$$\Delta t = \frac{\rho^{(1)}(t, x)}{\dot{\tilde{\rho}}(t)}. \quad (\text{C.8})$$

We conclude that our definition of the second order $\tilde{\zeta}_i^{(2)1}$ in the uniform energy density gauge agrees with the gauge-invariant quantities defined by other authors.

C.2 Gradients and locality

As a consistency check we want to verify that $\tilde{\zeta}_i^{(2)1}$ is indeed a total gradient, as it should be according to the first part of this appendix and (2.30). Taking expression (2.19) (corrected by the gauge transformation),

$$\begin{aligned} \tilde{\zeta}_i^{(2)1} = & -(\partial_i v_{e*})v_{f*} \left\{ -2\eta^\perp G_{1f}(t, t_*)G_{2e}(t, t_*) + \int_{t_*}^t dt' G_{1a}(t, t') \bar{A}_{abc} G_{be}(t', t_*) G_{cf}(t', t_*) \right\} \\ & + G_{1a}(t, t_*) L_{aef*} \partial_i (v_{e*} v_{f*}) + G_{1a}(t, t_*) N_{aef*} (\partial_i v_{e*}) v_{f*}, \end{aligned} \quad (\text{C.9})$$

and rewriting it using (D.1) we find

$$\begin{aligned} \tilde{\zeta}_i^{(2)1} = & -(\partial_i v_{e*})v_{f*} \left\{ A_{ab}(t_*) G_{1a}(t, t_*) \delta_{be} \delta_{1f} + \int_{t_*}^t dt' G_{1a}(t, t') \bar{A}_{abc} G_{be}(t', t_*) G_{cf}(t', t_*) \right. \\ & \left. - \int_{t_*}^t dt' A_{ab} A_{1c} G_{1a}(t, t') G_{be}(t', t_*) G_{cf}(t', t_*) - G_{1a}(t, t_*) N_{aef*} \right\} \\ & + G_{1a}(t, t_*) L_{aef*} \partial_i (v_{e*} v_{f*}). \end{aligned} \quad (\text{C.10})$$

This expression should be symmetrical under the interchange of the indices e and f . Notice that the last term is automatically symmetrical.

The anti-symmetrical part of the two integrands turns out to be proportional to

$$T_a = G_{23}(t', t_*) G_{32}(t', t_*) - G_{22}(t', t_*) G_{33}(t', t_*). \quad (\text{C.11})$$

We explicitly check the exact numerical value of this quantity and find it to be zero. For this it is crucial that we have defined t_* as the time a few (about 3) e-folds after horizon crossing. The reason is that the long-wavelength approximation we use in all our derivations is only valid once the rapidly decaying mode can be neglected, which takes a few e-folds. If the above quantity were to be evaluated before that time, it would not yet be zero. Note that in the slow-roll case T_a is identically zero according to (4.2), since within the slow-roll approximation the decaying mode is neglected by construction. The above means that $\tilde{\zeta}_i^{(2)1}$ is well defined only after it is well outside the horizon, where we can neglect the decaying mode. The case where the decaying mode can remain important is treated in the one-field case in [68].

The remaining non-integral terms between the braces can be explicitly checked to cancel when taking the slow-roll limit at horizon crossing: the first term of the first line of (C.10) gives

$$-(\partial_i v_{e*})v_{f*} A_{ab}(t_*) G_{1a}(t, t_*) \delta_{be} \delta_{1f} = \left(2\eta^\perp - \chi G_{12}(t, t_*) \right) (v_{1*} \partial_i v_{2*} - v_{2*} \partial_i v_{1*}), \quad (\text{C.12})$$

while the terms arising from the non-local contribution N_{aef*} gives exactly the same but with opposite sign. Hence we see that within the conditions of the long-wavelength approximation $\tilde{\zeta}_i^{(2)1}$ is indeed a total gradient, as it should be.

D Detailed calculations

D.1 Relation between space and time derivatives

We begin by proving that

$$\bar{A}_{ab1} = -\frac{1}{NH} \partial_t A_{ab} \quad (\text{D.1})$$

for all gauges with $\partial_i \ln a = 0$. Actually the statement is more general: the prefactor of ζ_i^1 in the expression for $\partial_i f$, where f is any function of $H, \phi, \mathbf{\Pi}$, is equal to $-(\partial_t f)/(NH)$ for gauges satisfying $\partial_i \ln a = 0$.

We start by computing the time derivative of f :

$$\begin{aligned} \frac{\partial_t f(H, \phi, \mathbf{\Pi})}{NH} &= \frac{1}{NH} (\partial_H f \partial_t H + \nabla_\phi f \cdot \partial_t \phi + \nabla_{\mathbf{\Pi}} f \cdot \partial_t \mathbf{\Pi}) \\ &= -H\epsilon \partial_H f + \frac{\sqrt{2}\epsilon}{\kappa} \mathbf{e}_1 \cdot \nabla_\phi f + \frac{\sqrt{2}\epsilon}{\kappa} H \boldsymbol{\eta} \cdot \nabla_{\mathbf{\Pi}} f, \end{aligned} \quad (\text{D.2})$$

where we used the definitions of the slow-roll parameters ϵ and $\boldsymbol{\eta}$, and of the basis vector \mathbf{e}_1 . On the other hand, the spatial derivative of f , in a $\partial_i \ln a = 0$ gauge, is given by

$$\begin{aligned} \partial_i f(H, \phi, \mathbf{\Pi}) &= \partial_H f \partial_i H + \nabla_\phi f \cdot \partial_i \phi + \nabla_{\mathbf{\Pi}} f \cdot \partial_i \mathbf{\Pi} \\ &= H\epsilon \partial_H f \mathbf{e}_1 \cdot \boldsymbol{\zeta}_i - \frac{\sqrt{2}\epsilon}{\kappa} \boldsymbol{\zeta}_i \cdot \nabla_\phi f \\ &\quad - \frac{\sqrt{2}\epsilon}{\kappa} H \left(\frac{1}{NH} \boldsymbol{\theta}_i + (\epsilon + \eta^\parallel) \boldsymbol{\zeta}_i - \epsilon (\mathbf{e}_1 \cdot \boldsymbol{\zeta}_i) \mathbf{e}_1 \right) \cdot \nabla_{\mathbf{\Pi}} f, \end{aligned} \quad (\text{D.3})$$

using the constraint relations for $\partial_i H$, $\partial_i \phi$, and $\partial_i \mathbf{\Pi}$ given in [6, 12]. (Note that some time and space derivatives have to be replaced with their covariant (in field space) version to make contact with the more general expressions given in those papers that take into account a non-trivial field metric.) Taking the components of $\boldsymbol{\zeta}$ and $\boldsymbol{\theta}$ in the basis defined in A, not forgetting the relation $\mathbf{e}_m \cdot \boldsymbol{\theta}_i = \theta_i^m + NH Z_{mn} \zeta_i^n$, with $Z_{21} = \eta^\perp$ [6, 12], we prove the stated relation, of which (D.1) is a special case.

D.2 Derivation of equation (3.6)

In this appendix we work out the last term of (3.2), which has to be added to the result for the second term (I) given in (3.5), to derive the final expression (3.6) for f_{NL} . We call the sum of these two terms J :

$$\frac{J}{\gamma_*^2} \equiv \frac{I}{\gamma_*^2} + \int_{t_*}^t dt' G_{1a} \bar{A}_{abc} \bar{v}_{bm} \bar{v}_{cn}, \quad (\text{D.4})$$

which is

$$\begin{aligned} \frac{J}{\gamma_*^2} &= \delta_{m2} \delta_{n1} \left(-2\eta_*^\perp + \chi_* G_{12}(t, t_*) + A_{32*} G_{13}(t, t_*) - \chi_* A_{33*} G_{13}(t, t_*) \right) \\ &\quad - \delta_{m2} \delta_{n2} 2\eta_*^\perp \chi_* G_{13}(t, t_*) \\ &\quad + \delta_{m2} \delta_{n2} \int_{t_*}^t dt' \left[\left(\bar{A}_{122} - 4(\eta^\perp)^2 + \bar{A}_{322} G_{13} \right) (\bar{v}_{22})^2 + \left(\bar{A}_{333} + 2\eta^\perp \right) G_{13} (\bar{v}_{32})^2 \right. \\ &\quad \left. + \left(\bar{A}_{123} - 4\eta^\perp G_{12} + (\bar{A}_{323} + \bar{A}_{332} + 2\eta^\perp A_{33} + 2\dot{\eta}^\perp) G_{13} \right) \bar{v}_{22} \bar{v}_{32} \right]. \end{aligned} \quad (\text{D.5})$$

We remind the reader that the bar on top of an index (\bar{c}) means that it does not take the value 1 and that a subscript $*$ means that a quantity is evaluated at t_* . The explicit form of

the matrix $\bar{\mathbf{A}}$ is given in [6]:

$$\begin{aligned}
\bar{A}_{121} &= 2\epsilon\eta^\perp - 4\eta^\parallel\eta^\perp + 2\xi^\perp, \\
\bar{A}_{122} &= -6\chi - 2\epsilon\eta^\parallel - 2(\eta^\parallel)^2 - 2(\eta^\perp)^2, \\
\bar{A}_{123} &= -6 - 2\eta^\parallel, \\
\bar{A}_{321} &= -12\epsilon\eta^\parallel - 12(\eta^\perp)^2 - 6\epsilon\chi - 8\epsilon^3 - 20\epsilon^2\eta^\parallel - 4\epsilon(\eta^\parallel)^2 - 12\epsilon(\eta^\perp)^2 \\
&\quad + 16\eta^\parallel(\eta^\perp)^2 - 6\epsilon\xi^\parallel - 12\eta^\perp\xi^\perp + 3(\tilde{W}_{111} - \tilde{W}_{221}), \\
\bar{A}_{322} &= -24\epsilon\eta^\perp - 12\eta^\parallel\eta^\perp + 24\eta^\perp\chi - 12\epsilon^2\eta^\perp + 8(\eta^\parallel)^2\eta^\perp + 8(\eta^\perp)^3 \\
&\quad - 8\epsilon\xi^\perp - 4\eta^\parallel\xi^\perp + 3(\tilde{W}_{211} - \tilde{W}_{222}), \\
\bar{A}_{323} &= 12\eta^\perp - 4\epsilon\eta^\perp + 8\eta^\parallel\eta^\perp - 4\xi^\perp, \\
\bar{A}_{331} &= -2\epsilon^2 - 4\epsilon\eta^\parallel + 2(\eta^\parallel)^2 - 2(\eta^\perp)^2 - 2\xi^\parallel, \\
\bar{A}_{332} &= -4\epsilon\eta^\perp - 2\xi^\perp, \\
\bar{A}_{333} &= -2\eta^\perp,
\end{aligned} \tag{D.6}$$

while the rest of the matrix elements are zero. Using these expressions we have

$$\begin{aligned}
\bar{A}_{333} + 2\eta^\perp &= 0, \\
\bar{A}_{323} + \bar{A}_{332} + 2\eta^\perp A_{33} + 2\dot{\eta}^\perp &= 18\eta^\perp - 4\dot{\eta}^\perp, \\
\bar{A}_{123} &= -2A_{33} + 2\epsilon + 2\eta^\parallel, \\
\bar{A}_{122} - 4(\eta^\perp)^2 &= -2A_{32} + 2\dot{\epsilon} + 2\dot{\eta}^\parallel,
\end{aligned} \tag{D.7}$$

so that we can write

$$\begin{aligned}
\frac{J}{\gamma_*^2} &= \delta_{m2}\delta_{n1} \left(-2\eta_*^\perp + \chi_* G_{12}(t, t_*) + A_{32*} G_{13}(t, t_*) - \chi_* A_{33*} G_{13}(t, t_*) \right) \\
&\quad - \delta_{m2}\delta_{n2} 2\eta_*^\perp \chi_* G_{13}(t, t_*) \\
&\quad + \delta_{m2}\delta_{n2} \int_{t_*}^t dt' \left[2(\bar{v}_{22})^2 \frac{d}{dt'} (\epsilon + \eta^\parallel) + 2\bar{v}_{22} \frac{d}{dt'} \bar{v}_{32} - 4 \left(\eta^\perp G_{12} + \dot{\eta}^\perp G_{13} \right) \frac{1}{2} \frac{d}{dt'} (\bar{v}_{22})^2 \right. \\
&\quad \left. + 2(\epsilon + \eta^\parallel) \bar{v}_{22} \bar{v}_{32} + \bar{A}_{322} G_{13} (\bar{v}_{22})^2 + 18\eta^\perp G_{13} \bar{v}_{22} \bar{v}_{32} \right].
\end{aligned} \tag{D.8}$$

Doing integrations by parts on the three terms in the third line we obtain

$$\begin{aligned}
\frac{J}{\gamma_*^2} &= \delta_{m2}\delta_{n1} \left(-2\eta_*^\perp + \chi_* G_{12}(t, t_*) + A_{32*} G_{13}(t, t_*) - \chi_* A_{33*} G_{13}(t, t_*) \right) \\
&\quad + 2\delta_{m2}\delta_{n2} \left(-\eta_*^\perp \chi_* G_{13}(t, t_*) - (\epsilon_* + \eta_*^\parallel) + \chi_* + \eta_*^\perp G_{12}(t, t_*) \right. \\
&\quad \left. + \dot{\eta}_*^\perp G_{13}(t, t_*) + (\epsilon + \eta^\parallel) (\bar{v}_{22})^2 + \bar{v}_{22} \bar{v}_{32} \right) \\
&\quad + 2\delta_{m2}\delta_{n2} \int_{t_*}^t dt' \left[-2(\eta^\perp)^2 (\bar{v}_{22})^2 - (\epsilon + \eta^\parallel) \bar{v}_{22} \bar{v}_{32} - (\bar{v}_{32})^2 + 9\eta^\perp G_{13} \bar{v}_{22} \bar{v}_{32} \right. \\
&\quad \left. + \frac{1}{2} \left(\bar{A}_{322} + 2\ddot{\eta}^\perp + 2\dot{\eta}^\perp A_{33} + 2\eta^\perp A_{32} \right) G_{13} (\bar{v}_{22})^2 \right].
\end{aligned} \tag{D.9}$$

The following relation (derived by taking two time derivatives of the field equation) can be used to remove higher-order slow-roll parameters:

$$\tilde{W}_{m11} = -\frac{\eta_m^{(4)}}{3} - \left(1 - \frac{\eta^\parallel}{3} \right) \xi_m + (2\epsilon + \eta^\parallel) \eta_m + \epsilon \eta^\parallel \delta_{m1} - \eta^\perp \tilde{W}_{m2}. \tag{D.10}$$

Explicitly, for $m = 1$ and $m = 2$ in the case of two fields, this becomes

$$\begin{aligned}\tilde{W}_{111} &= -\frac{1}{3}\eta^{(4)\parallel} - \left(1 - \frac{1}{3}\eta^{\parallel}\right)\xi^{\parallel} + 3\epsilon\eta^{\parallel} + (\eta^{\parallel})^2 + (\eta^{\perp})^2 + \frac{1}{3}\eta^{\perp}\xi^{\perp}, \\ \tilde{W}_{211} &= -\frac{1}{3}\eta^{(4)\perp} - \left(1 - \frac{1}{3}\eta^{\parallel}\right)\xi^{\perp} + 3\epsilon\eta^{\perp} + 2\eta^{\parallel}\eta^{\perp} - \eta^{\perp}\chi.\end{aligned}\quad (\text{D.11})$$

Using the second of these relations, as well as the explicit expression for \bar{A}_{322} , we find that

$$\begin{aligned}\bar{A}_{322} + 2\ddot{\eta}^{\perp} + 2\dot{\eta}^{\perp}A_{33} + 2\eta^{\perp}A_{32} \\ = 24\eta^{\perp}\chi - 12\eta^{\parallel}\eta^{\perp} + 12(\eta^{\parallel})^2\eta^{\perp} + 12(\eta^{\perp})^3 - 4\eta^{\perp}\xi^{\parallel} - 4\eta^{\parallel}\xi^{\perp} - 3(\tilde{W}_{211} + \tilde{W}_{222}).\end{aligned}\quad (\text{D.12})$$

We now drop boundary terms that are second order in the slow-roll parameters *at horizon crossing*, since it would be inconsistent to include them given that the linear solutions used at horizon crossing are only given up to first order. Then the result is

$$\begin{aligned}\frac{J}{\gamma_*^2} &= \delta_{m2}\delta_{n1}(-2\eta_*^{\perp} + \chi_* \bar{v}_{12}) + 2\delta_{m2}\delta_{n2}\left(-\epsilon_* - \eta_*^{\parallel} + \chi_* + \eta_*^{\perp}\bar{v}_{12} + (\epsilon + \eta^{\parallel})(\bar{v}_{22})^2 + \bar{v}_{22}\bar{v}_{32}\right) \\ &\quad + 2\delta_{m2}\delta_{n2}\int_{t_*}^t dt' \left[-2(\eta^{\perp})^2(\bar{v}_{22})^2 - (\epsilon + \eta^{\parallel})\bar{v}_{22}\bar{v}_{32} - (\bar{v}_{32})^2 + 9\eta^{\perp}G_{13}\bar{v}_{22}\bar{v}_{32} \right. \\ &\quad \left. + \left(12\eta^{\perp}\chi - 6\eta^{\parallel}\eta^{\perp} + 6(\eta^{\parallel})^2\eta^{\perp} + 6(\eta^{\perp})^3 - 2\eta^{\perp}\xi^{\parallel} - 2\eta^{\parallel}\xi^{\perp} \right. \right. \\ &\quad \left. \left. - \frac{3}{2}(\tilde{W}_{211} + \tilde{W}_{222})\right)G_{13}(\bar{v}_{22})^2 \right].\end{aligned}\quad (\text{D.13})$$

Inserting this into (3.1) gives the final result for f_{NL} in (3.6).

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